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Estimating the Lyapunov-Exponent Spectrum from Short Time Series of Low Precision

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We propose a new method to compute Lyapunov exponents from limited experimental data. The method is tested on a variety of known model systems, and it is found that the algorithm can be used to obtain a reasonable Lyapunov-exponent spectrum from only 5000 data points with a precision of 10^{-1} or 10^{-2} in three- or four-dimensional phase space, or 10000 data points in five-dimensional phase space. We also apply our algorithm to the daily-averaged data of surface temperature observed at two locations in the United States to quantitatively evaluate atmospheric predictability.

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Nonlinear phenomena occur in nature in a wide range of apparently different contexts, yet they often display common features, or can be understood using similar concepts. Deterministic chaos and fractal structure in dissipative dynamical systems are among the most important nonlinear paradigms. The spectrum of Lyapunov exponents provides a quantitative measure of the sensitivity to initial conditions (i.e., the divergence of neighboring trajectories exponentially in time) and is the most useful dynamical diagnostic for chaotic systems. In fact, any system containing at least one positive Lyapunov exponent is defined to be chaotic, with the magnitude of the exponent determining the time scale for predictability. In any well-behaved dissipative dynamical system, one of the Lyapunov exponents must be strictly negative.[1] If the Lyapunov-exponent spectrum can be determined, the Kolmogorov entropy[2] can be computed by summing all of the positive exponents, and the fractal dimension may be estimated using the Kaplan-Yorke conjecture.[3]

The Lyapunov-exponent spectrum can be computed relatively easily for known model systems.[4] However, it is difficult to estimate Lyapunov exponents from experimental data for a complex system (e.g., the atmosphere). Wolf *et al.*[5] proposed a method to estimate one or two positive exponents. Sano and Sawada[6] and Eckmann *et al.*[7] developed similar procedures to determine several of the Lyapunov exponents (including positive, zero, and even negative values). This is now a very active research area, and several authors[8] have introduced further improvements. However, all of these methods require relatively long time series and/or data of high precision (for example, Eckmann *et al.* used 64000 data points with a precision of 10^{-4} for the Lorenz equations[9]), but such high-quality data cannot be obtained in many real-world situations.

The infinitesimal length scales used to define Lyapunov exponents are inaccessible in experimental data.[5] The presence of noise or limited precision leads to a length scale L_n below which the structure of the underlying strange attractor is obscured. Also, for a finite data set of N points, there is a minimum length scale $L_0 = L/N^{1/D}$, where L is the horizontal extent of the attractor and D is its information dimension,[10] below which structure cannot be resolved. When $L_0 \leq L_n$, increasing N is not likely to provide any further information on the structure of the attractor, so that a relatively small data set can be sufficient for computing Lyapunov exponents. Furthermore, if the length scales L_0 and L_n are small enough for the chaotic dynamics to be the same as at infinitesimal length scales, then the computation of Lyapunov exponents using these length scales should yield reasonable results.

Abraham *et al.*[11] have demonstrated that it is possible to calculate the dimensions of attractors from small, noisy data sets. The purpose of this paper is to develop a procedure by which one can evaluate the Lyapunov-exponent spectrum from relatively small data sets of low precision. We test the method on a variety of known model systems, and we also use the method to study the predictability of the atmosphere from observational meteorological data. It should be noted, as pointed out by Ruelle,[12] that the Grassberger-Procaccia algorithm[13] cannot be used for small data sets, but that no such restrictions apply to Lyapunov exponents and the Kaplan-Yorke dimension.

Given a time series $x_i = x(i\Delta t)$ $(i = 1,2,\ldots,N)$, where N is the number of observations and Δt is the time interval between measurements, the attractor can be reconstructed in a k-dimensional phase space[14] by forming the vectors

$$\mathbf{x}_i = (x_i, x_{i+m}, \ldots, x_{i+(k-1)m}),$$

where $\tau = m\Delta t$ is the time delay, with the integer m chosen appropriately. Different methods have been suggested to obtain τ (see Zeng, Pielke, and Eykholt[15] for detailed discussions). In this paper, τ is chosen as the lag time at which the autocorrelation function of the time series falls to $e^{-1} \approx 0.37$.

For each point $\mathbf{x}_i$, consider the shell between two spheres centered at $\mathbf{x}_i$ of radii $r_{\min} < r$, and consider the set of trajectory points $\mathbf{x}_j$ within this ith shell:

$$r_{\min} \le \|\mathbf{x}_j - \mathbf{x}_i\| = \left[\sum_{l=0}^{k-1} (x_{j+lm} - x_{i+lm})^2\right]^{1/2} \le r.$$

The use of a shell, rather than a ball, is to minimize the effects of noise or measurement error, since these effects are greatest when $\|\mathbf{x}_j - \mathbf{x}_i\|$ is small. After a time $n\Delta t$, the small vectors $\mathbf{x}_j - \mathbf{x}_i$ evolve to the small vectors $\mathbf{x}_{j+n} - \mathbf{x}_{i+n}$. If these vectors are so small that they can be regarded as good approximations to tangent vectors in the tangent space of the dynamical system, a $k \times k$ matrix T_i describing the evolution can be obtained from the equations

$$\mathbf{x}_{j+n} - \mathbf{x}_{i+n} = T_i(\mathbf{x}_j - \mathbf{x}_i). \tag{1}$$

The elements of the matrix T_i are found using a least-squares-error algorithm.[6] In the special case $n = m$ (i.e., $n\Delta t = m\Delta t = \tau$), the matrix T_i consists of 1's just above the diagonal and 0's elsewhere, except for the last row of elements. Our computations have shown that results using $n = m$ are usually as good as, or even better than, those for $n < m$, and computations with $n = m$ are much less time consuming, so we use $n = m$ in our calculations below.

When the number n_i of points in the ith shell is not less than the embedding dimension k, the algorithm succeeds most of the time. However, to be conservative and reduce statistical errors, we use only those shells for which n_i is much larger than k (in the computations below, n_i is taken to be 10). We first take r to be 5% of the horizontal extent L of the attractor, since Eq. (1) requires $\mathbf{x}_j - \mathbf{x}_i$ to be small. In experimental data, this generally makes n_i sufficiently large, and the noise length scale is generally less than r. In the case that some n_i is too small, we double r to $0.1L$ for that shell and find the trajectory points $\mathbf{x}_j$ within this new shell, although this is seldom necessary. If n_i is still too small, we drop this point $\mathbf{x}_i$ and proceed to the next point $\mathbf{x}_{i+n}$. We take $r_{\min}$ to be the length scale of the noise, which, in our examples, is $0.01L$. The matrix T_i is successively reorthogonalized by means of a standard Q_iR_i decomposition.[16] Then the Lyapunov exponents are given by[7]

$$\lambda_l = \frac{1}{m\Delta t K} \sum_{j=0}^{K-1} \ln(R_j)_{ll}, \quad l = 1,2,\ldots,k,$$

where $K \le [N - (k-1)m - 1]/n$ is the available number of matrices T_i.

The x components of numerical data for various known model systems are treated as experimental data to test our algorithm. These systems are the Lorenz equations[9] and the Rossler equations,[17] which are finite-dimensional systems, and the Mackey-Glass equations,[18] which constitute an infinite-dimensional system. The first two systems are solved by the Runge-Kutta method, and the last system is solved by a very efficient algorithm of second-order precision.[19] We use a time step $\Delta t = 0.01$ for the Lorenz equations and $\Delta t = 0.1$ for the Rossler equations. A time step of $0.01T$, where the parameter T is given in Table I, is used to integrate the Mackey-Glass equations. However, we then include only every fifth value in our data set, producing a time series with $\Delta t = 0.05T$, so that the delay time τ is not too large compared with Δt (usually, $\tau \approx 10\Delta t$ is desired[20]).

The first 10000 data are discarded from the generated time series to eliminate transients, and the number N of observations is taken to be 5000, except for the Mackey-Glass equations with $T = 30$, for which a five-dimensional phase space is used, and we take $N = 10000$. For the Lorenz and Rossler equations, all values are rounded off to the first decimal, producing a precision of 10^{-1}, and for the Mackey-Glass equations, all values are rounded off to a precision of 10^{-2} (this is because the horizontal extent of the attractor is much smaller in this case). We take $K = \min(2000, [N - (k-1)m - 1]/m)$ to guarantee saturated Lyapunov exponents, although convergence of λ_i is actually reached with fewer matrices (Fig. 1 shows the convergence of λ_i for the Mackey-Glass equations). The autocorrelation function is also illustrated in Fig. 1, and it is seen that the delay time τ (i.e., the e-folding time of the autocorrelation curve) is about $9\Delta t$.

Table I shows the computed Lyapunov-exponent spectrum for the various model systems described above. The error bars are computed from a few runs with changes in the parameters τ, $r_{\min}$, and r. It is seen that all error bars are relatively small, which shows that the result from our algorithm are insensitive to the choice of these parameters. For the Lorenz equations, the computed value of the largest positive Lyapunov exponent λ_1 differs from the accepted value by less than 9%. Since the value obtained for λ_2 is only about 3% of λ_1, its relative error is very large. However, one exponent must be zero, and this exponent is easily identified as λ_2, so that the relative error for λ_2 has little meaning. For the Rossler equations, λ_1 is obtained with a relative error less than 7%, and λ_2 is less than 7% of λ_1. For the Mackey-

TABLE I. Lyapunov-exponent spectrum for various known model systems. The parameters used in the different systems, the total number of data points N, the precision of the data (from 10^{-1} to 10^{-4}), and the delay time τ are given in the table; all other parameters are as described in the text.

System	Reported λ_i (in the absence of noise)	Computed λ_i (in the presence of noise)
Lorenz ($\tau = 20\Delta t$)	1.50 (Ref. 5)	1.63 ± 0.15
($\sigma = 16$, $b = 4.0$, $R = 45.92$)	0.00	0.05 ± 0.25
($N = 5000$, 10^{-1} precision)	-22.46	-3.59 ± 0.41
Rossler ($\tau = 12\Delta t$)	0.090 (Ref. 5)	0.096 ± 0.008
($a = 0.15$, $b = 0.2$, $c = 10$)	0.00	-0.006 ± 0.004
($N = 5000$, 10^{-1} precision)	-9.8	-0.735 ± 0.057
Mackey-Glass ($\tau = 9\Delta t$)	0.0071 (Ref. 6)	0.0075 ± 0.0007
($a = 0.2$, $b = 0.1$, $c = 10$, $T = 30$)	0.0027	0.0030 ± 0.0010
($N = 10,000$, 10^{-2} precision)	0.000	-0.0027 ± 0.0010
	-0.0167	-0.0156 ± 0.0006
	-0.0245	-0.0394 ± 0.0064
Mackey-Glass ($\tau = 9\Delta t$)	0.00956 ± 0.00005 (Ref. 21)	0.00938 ± 0.00040
($a = 0.2$, $b = 0.1$, $c = 10$, $T = 23$)	0.00000	0.00008 ± 0.00020
($N = 5000$, 10^{-2} precision)	-0.0119 ± 0.0001	-0.0160 ± 0.0010
	-0.0344 ± 0.0001	-0.0734 ± 0.0227
Mackey-Glass ($\tau = 9\Delta t$)	0.00956 ± 0.00005 (Ref. 21)	0.00946 ± 0.00008
($a = 0.2$, $b = 0.1$, $c = 10$, $T = 23$)	0.00000	0.00064 ± 0.00049
($N = 30,000$, 10^{-4} precision)	-0.0119 ± 0.0001	-0.0134 ± 0.0011
	-0.0344 ± 0.0001	-0.0572 ± 0.0135

Glass equations with $T=23$ and only 5000 data points, λ_1 is obtained with a relative error less than 2%, and λ_2 is less than 1% of λ_1. For the Mackey-Glass system with $T=30$, a five-dimensional phase space is used, requiring 10000 data points, rather than 5000, so that the density of data points defining the attractor is still acceptable.

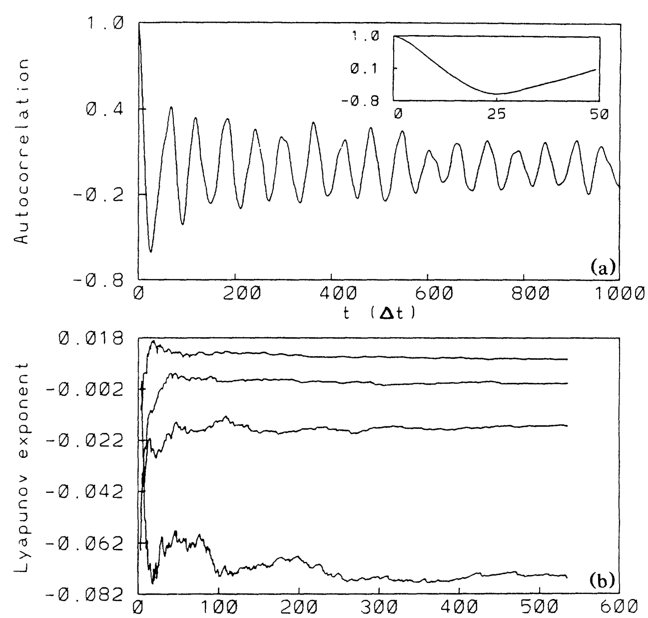

FIG. 1. (a) Autocorrelation function, and (b) convergence of Lyapunov exponents for the Mackey-Glass equations with parameters $a=0.2$, $b=0.1$, $c=10$, and $T=23$, and other parameters as described in the text. The inset graph is a magnification of the region close to the origin in (a).

In this case, λ_1 is obtained with a relative error less than 6%, and the second positive exponent λ_2 is also obtained with a relative error of only about 11%. When data of higher precision were used, much smaller relative errors were obtained; however, given the low precision of these data (i.e., the high noise level), better agreement with the values in the absence of noise is not to be expected.

The possibility of obtaining reasonable negative Lyapunov exponents depends on their magnitudes and the signal-to-noise ratio of the data.[6] Since a precision of 10^{-1} or 10^{-2} is prescribed (i.e., the signal-to-noise ratio of the data is low), and $|\lambda_3|$ is more than a hundred times larger than λ_1 for the Rossler equations, the computed $|\lambda_3|$ is too small compared with the reported $|\lambda_3|$. However, when the absolute values of the negative exponents are comparable with λ_1, as for the Mackey-Glass equations with $T=30$ or 23, we obtain negative exponents which are comparable to the reported values.

Therefore, using various known model systems, both finite and infinite dimensional, we have shown that our algorithm can be used to evaluate the Lyapunov-exponent spectrum from only 5000 data points of very low precision (10^{-1} or 10^{-2}) in a phase space whose dimension is less than 5, and from 10000 points of low precision in five-dimensional phase space. Since the number and precision requirements for observational data are often of this order, and since no adjusting of free parameters is needed, our algorithm is particularly easy to apply and may find widespread applications in practice.

Noise is an infinite-dimensional process and will tend to decrease the density of points defining the attractor as the embedding dimension k increases.[5] Since the way we obtain τ guarantees linear independence, the mini-

TABLE II. Lyapunov spectrum λ_i and the error-doubling time T computed from measurements of the daily surface temperature at LA and FCL.

Location	LA	FCL
parameters	$\Delta t = 1$ day $\tau = 4$ days $N = 14245$	$\Delta t = 1$ day $\tau = 3$ days $N = 36555$
λ_i	0.121 0.065 0.004 -0.059 -0.174	0.195 0.081 0.016 -0.077 -0.220
T (day)	3.7	2.5

mally required k (e.g., $k=3$ for the Lorenz equations) often yields reasonable Lyapunov exponents and greatly reduces computer time. This has been confirmed in our computations using different values of k. On the other hand, when τ is too small (e.g., $\tau=\Delta t=0.03$, or $m=1$, in Eckmann *et al.*[7]), $x_i, x_{i+m}, \ldots, x_{i+(k-1)m}$ are not independent, and the minimally required k leads to a phase space of dimension less than k. This explains why $k=3$ for the Lorenz equations did not yield good results in Eckmann *et al.*[7] With their method, k must be increased, which requires increasing the number of data points and their precision so that the level of contamination of the data remains relatively low. The use of d_E and d_M in Eckmann *et al.*[7] plays a role similar to increasing the delay time τ.

Our algorithm has been applied to daily observational data of temperature and pressure over the United States and the North Atlantic Ocean. Detailed results will be published elsewhere,[15] but we briefly summarize them here. The Lyapunov-exponent spectrum computed from the time series of surface temperature in Los Angeles (LA), California, and at Fort Collins (FCL), Colorado, are shown in Table II. Since one of the exponents must be zero, we recognize that $\lambda_3=0$ (which is well within the error bars). Thus, the sum of the two positive Lyapunov exponents gives an estimate of the Kolmogorov entropy, and its inverse, multiplied by ln2, gives the predictability (error-doubling) time T, which is also shown in Table II. It is seen that the time series for the temperature has two positive Lyapunov exponents, which implies that the atmosphere has a hyperchaotic attractor, with an error-doubling time T of about 3.7 days in LA, where the climatic signal-to-noise ratio is high, and about 2.5 days in FCL, where the signal-to-noise ratio is relatively low. These values are within the range of previous estimates.[22] Therefore, our method offers a new way to study atmospheric predictability quantitatively, which is superior to the traditional, qualitative, signal-to-noise analyses.

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