Anderson Replies: I am grateful to Randeria and Engelbrecht[1] for an opportunity to expand on my sketchy discussion.

A main difficulty here is a confusion in terms between two quantities: the scattering phase *shift* Θ_{sc} and the phase *angle* Θ_Γ of the conventional particle-particle vertex, a function of energy which Engelbrecht and Randeria calculate correctly. These are independent quantities and have different physical meanings. What I calculated in the Letter is the scattering phase *shift*, a number which determines the boundary condition at the origin in any scattering event between two particles of opposite spin in the Fermi sea at momenta k and k'. It characterizes the pseudopotential for scattering and describes the local wave-function modification when particles in the Fermi sea impinge on each other. Thus it is independent of the treatment of poles and imaginary parts on the energy shell. In the simple calculation I did, the scattering phase shift was determined by the principal-part integral

$$P(E)=P\int dE_Q\frac{n(E_Q)(1-f_{k+Q})(1-f_{k-Q})}{E-E_Q}\,, \quad (1)$$

and the way the phase shift was determined was equivalent to

$$\theta_{sc}=\arg\frac{U}{1+U[i\pi n(E)+P(E)]}\,. \quad (2)$$

The $i\pi n(E)$ term does *not* represent an imaginary part of any pair Green's function χ, but is just the formal expression of outgoing boundary conditions for the scattering. This phase shift does not vanish for most particle-particle scatterings. It does vanish for forward scattering in >1D if unlimited recoil is allowed. It is the phase shift to which Friedel's theorem applies: $-\delta/\pi$ tells us how many extra particles are contained in the wave functions in the region of the scattering, or, equivalently, it tells us the momentum shift of the wave functions. For particles below the Fermi energy, it can tell us how the "sea" of occupied states for spin down is deformed by the presence of a spin-up particle, an effect which requires great care to include in conventional perturbation theory[2] (see below).

A separate theorem applies to the phase angle Θ_Γ of the Comment. This tells us of any bound states or resonances in the *quasiparticle* spectrum due to the *residual* interactions, but it contains no information about Fermi-liquid parameters, since the residual interactions are irrelevant in the sense of renormalization theory. Fermi-liquid parameters cannot be deduced from the usual coupling-constant integration because the effects are not analytic in U. They must be found by explicitly solving an appropriate Bethe-Salpeter equation for two real holes of specified momenta. This could be done diagrammatically by reinserting the "anomalous" terms[2] representing the on-energy shell scattering *with appropriate treatment of boundary conditions* at ∞, which brings in terms mixing $-k$ hole (incoming) with $+k$ electron (outgoing) states, among others.

It is much easier, however, to simply solve a Schrödinger equation in a finite system for a real up-spin particle of fixed k scattering a down-spin particle k'. All virtual scattering events are modeled by the phase shift δ_{sc}, so that we are only concerned to find the asymptotic wave function in the relative coordinates of the two holes. This is simply ($Q=k'-k$)

$$(r_1-r_2)^{-1/2}\cos[Q(r_1-r_2)+\delta_{sc}]\,. \quad (3)$$

This means that $Q_{l=0}$ for the isotropic partial wave must be modified by

$$\delta Q=\delta_{sc}/\pi R \quad (4)$$

to satisfy boundary conditions. This is, however, only one Q out of $2QR$ partial waves, so that the *average* shift is

$$\overline{\delta Q}=\delta/2QR^2 \quad (5)$$

in the radial direction. This form assures incompressibility of the Hilbert space of k values.

With $k_\uparrow$ fixed, this means that $k'_\downarrow$ must shift by $\overline{\delta Q}$ so that (with $\hbar=m=1$)

$$\Delta E=[\mathbf{k}'\cdot(\mathbf{k}'-\mathbf{k})/2(k'-k)^2R^2]\delta_{sc}\,. \quad (6)$$

Normalizing to unit volume, we find

$$f_{kk'}=\frac{\delta_{sc}}{\pi}\frac{\mathbf{k}'\cdot(\mathbf{k}'-\mathbf{k})}{2(k-k')^2} \quad (7)$$

which diverges as $Q\to 0$. Incompressibility of k space tells us that the total energy shift, which treating the state k as a hole excitation may be taken as the real part of a self-energy, is proportional to

$$\frac{\delta_{sc}}{\pi}|\epsilon_F-\epsilon_k|,\quad k<k_F\,;\;\; 0,\quad k>k_F\,. \quad (8)$$

Well-known Friedel theorem methods will guarantee that this result survives as $R\to\infty$, as we shift to outgoing boundary conditions for the Bethe-Salpeter equation.

I believe that (7) is the appropriate renormalized vertex to use in conventional many-body theory in which explicit anomalous forward-scattering diagrams and boundary conditions are ignored. Such a theory clearly has many interesting properties. This vertex is very like what might be expected for a "statistical," exclusionary interaction, as is not surprising since it arises from projective effects in Hilbert space.

Philip W. Anderson
Department of Physics
Princeton University
Princeton, New Jersey 08544

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[1]Jan R. Engelbrecht and Mohit Randeria, preceding Comment, Phys. Rev. Lett. **66**, 3225 (1991).

[2]W. Kohn and J. M. Luttinger, Phys. Rev. **118**, 41 (1960), first discussed this kind of effect in terms of the old Hubbard Bethe-Goldstone time-ordered perturbation theory, where it leads to the "anomalous" diagrams. These diagrams might appear to be automatically included in modern Feynman techniques, but are not.