Paired Hall State at Half Filling

Martin Greiter,[a] Xiao-Gang Wen, and Frank Wilczek
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540
(Received 4 April 1991)

The existence of a novel incompressible quantum liquid for spinless fermions at $\nu=\frac{1}{2}$ in the Hall effect is suggested. This state is plausibly related by smooth extrapolation in quantum statistics to a strong p-wave pairing state of fermions in zero magnetic field, and reduces to a state previously proposed by Halperin in the (unrealistic) limit of tightly bound pairs. It supports unusual excitations, including neutral fermions and charge-$e/4$ anyons with statistical parameter $\theta=\pi/8$. Numerical experiments are presented, which provide evidence for several aspects of our theory.

PACS numbers: 72.20.My, 05.30.Fk

Arguments for the state.—The possibility of anyon statistics allows one to consider perturbing or interpolating in particle statistics in two space dimensions.

Straightforward perturbation theory is likely to be problematic, however. Small changes in the statistics correspond to putting a small quantity of fictitious magnetic flux on each particle. Insofar as the important paths (in a path integral) wander over many interparticle spacings, they acquire their phase from winding around many tubes each individually responsible for only a small phase. Thus if we wish to treat the change in statistics as a small residual interaction, then if the paths wander—that is, in the regime of strong quantum phenomena, or low temperature—the proper starting point must be to analyze the problem in the appropriate background magnetic field.

This discussion is closely related to a heuristic principle recently emphasized by two of us.[1] Incompressible states along the lines

$$\Delta(\theta/\pi)=\Delta(1/\nu) \tag{1}$$

in the statistics–magnetic-field (or inverse filling fraction) plane are likely to be adiabatically related, by the procedure of trading flux localized on the particles for an equal amount of uniform flux. This procedure generalizes the successful RPA treatment of anyon superconductors at $\theta/\pi=1-1/n$ in zero field by perturbation around fermions ($\theta/\pi=1$) filling n Landau levels ($\nu=1/n$). It also relates, for $\Delta(\theta/\pi)=$even integer, the integer to fractional quantized Hall states of fermions—the connection being made through a continuous succession of anyon states. This observation forms the basis of a systematic perturbative approach to the latter states, which will be discussed at length elsewhere.

The states related to free fermions in *zero* magnetic field, that is anyons with statistical parameter $\theta=\pi(1+\epsilon)$ in a uniform magnetic field $B=2\pi\epsilon\rho/e$, are especially interesting. The reference problem at $\epsilon=0$ is poised on the brink of the BCS pairing instability, which can be triggered by arbitrarily weak interactions. Thus residual interactions may have a large effect even in the small-ϵ regime where their effects are reliably calculable. We find that the residual interactions do trigger a p-wave pairing instability at weak coupling, whose strength *increases* with ϵ. Thus although the perturbative calculation becomes unreliable, it still strongly suggests that this type of ordering occurs even for large ϵ.

One interesting point occurs at $\epsilon=1$. Here one has reached a state of *bosons* exactly filling one Landau level, $\nu=1$. There is some numerical evidence[2] that bosons with hard-core repulsion do in fact organize themselves into an incompressible state at $\nu=1$.

Traveling still further along the same line, at $\epsilon=2$ one arrives back at fermions. Thus fermions at filling fraction $\nu=\frac{1}{2}$ are related to fermions in zero magnetic field with residual local interactions that induce pairing.

Some qualitative consequences of pairing are most easily appreciated in the limit that the pairing is very strong. Following Halperin,[3] let us imagine that our fermions bind pairwise into effective bosons. $\nu=\frac{1}{2}$ for the fermions corresponds to $\nu=\frac{1}{8}$ for the bosons, since there are half as many particles each carrying twice the charge. For repulsive bosons at $\nu=\frac{1}{8}$ a conventional Laughlin wave function and corresponding incompressible state are available. The elementary charged excitations around this state are anyons with statistical parameter $\theta=\pi/8$ and carrying one-eighth of the boson charge, that is $e/4$.

Idealized ground-state wave functions.—There is an appealing variational wave function for fermions at $\nu=\frac{1}{2}$. It has appeared in unpublished work by Moore and Read[4] motivated by conformal field theory.

The wave function in question is of the form

$$\Psi_{1/2}=\mathrm{Pf}(1/(z_i-z_j))\psi_{1/2}\,; \tag{2}$$

$$\psi_{1/2}=\prod_{i<j}(z_i-z_j)^2\exp\left(-\frac{eB}{4}\sum_j|z_j|^2\right). \tag{3}$$

Equation (3) is the standard Laughlin state (suitable for *bosons*) at $\nu=\frac{1}{2}$ and Pf(M) is the Pfaffian[5] of the antisymmetric matrix M.

Dyson[6] pointed out long ago that the wave function for a BCS superconductor, written in real space for a definite number of particles, takes the form of a Pfaffian. Equation (3) can be obtained from the adiabatic evolution, following Refs. 7 and 8, of a singular superconducting state (for δ-function attraction) described by the Pfaffian factor alone.

Equation (2) is the exact ground state of a simple local effective Hamiltonian. The basic idea is most transparent for the paired $\nu=1$ boson state. Although it is not true that there is a zero in the wave function when any two particles coincide—the two particles in question might be paired—there is a zero whenever *three* particles coincide. Thus the pairing state is the ground state of a Hamiltonian with repulsive three-body δ function interactions.

$$V_{i;jk}=\sum_{\text{triples}}\delta^{(2)}(z_i-z_j)\delta^{(2)}(z_i-z_k) \tag{4}$$

annihilates the wave function. It is not difficult to demonstrate that the product of the Pfaffian and the full-Landau-level wave function is actually the lowest-degree polynomial annihilated by V. Similar constructions work for the paired Hall state at half filling.

For our later purposes it is important to consider the pairing state on a sphere. A wave function $\Psi(\{u_i,v_i\})$ in the first Landau level[9] must be homogeneous of degree N_ϕ in the spinor variables u_i and v_i, for every i. For the pairing wave function we have

$$\Psi=\mathrm{Pf}\left(\frac{1}{u_iv_j-v_iu_j}\right)\prod_{i<j}(u_iv_j-v_iu_j)^2 . \tag{5}$$

From the total power of u_i and v_i we infer the significant relation

$$N_\phi=2N-3 \tag{6}$$

between flux and particle number. It indicates that in the thermodynamic limit $N\to\infty$ we do in fact have a $\nu=\frac{1}{2}$ state. The finite shift by -3 flux quanta characterizes the internal correlations in the pairing state, and can serve as a signal for it in numerical experiments.

Equation (6) can also be derived from the tight-binding boson picture. Indeed if the bosons represent p-wave bound states of the electrons, they are spin-1 objects. The effective number of flux quanta visible to each boson is therefore composed of two pieces, $N_\phi^b=2N_\phi-2$, where the first piece is ordinary flux (times charge 2) and the shift -2 arises from the coupling of spin 1 to the curvature of the sphere. For the bosons to form a $\nu=\frac{1}{8}$ Laughlin state, their number $N^b=N/2$ must be related to the effective flux by $8(N^b-1)=N_\phi^b$; hence, Eq. (6).

Excitations.—Our paired state supports two classes of excitations that are not routine generalizations of those occurring in conventional odd-denominator states.

Several lines of argument suggest that the standard Laughlin construction for charged quasiparticles does not yield the most elementary charged excitations, rather the elementary charged excitations arise from insertion *of half* a flux quantum: (1) The genesis of the paired $\nu=\frac{1}{2}$ state from a paired BCS state. The mother BCS state certainly allows insertion of half a flux quantum, without disturbance of its long-range order. (2) The presumed connection to Halperin's strong-coupling effective boson quantized Hall state, which certainly allows excitations with the appropriate *electric charge* —i.e., $-e/4$. (3) The availability of a simple and attractive wave function embodying the possibility. (4) Finally and most convincing, the numerical experiments described below.

The wave function in question involves taking the Laughlin $\prod(z_i-\eta)$ factor inside the Pfaffian. Let η and ζ be two fixed positions. Then the wave function in question is generated by modifying the Pfaffian in the manner

$$\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)\to\mathrm{Pf}\left(\frac{(z_i-\eta)(z_j-\zeta)+(i\leftrightarrow j)}{z_i-z_j}\right). \tag{7}$$

There is no difficulty in writing similar wave functions on a sphere.

This wave function may also be motivated on more directly physical grounds. In the strong-binding limit, we may represent the pairs as effective bosons positioned at the center of mass of fermions. The insertion of Laughlin quasiholes *for the effective bosons* is implemented by a multiplicative factor

$$\prod_{\text{pairs}}\left(\frac{z_a+z_b}{2}-\eta\right)\left(\frac{z_a+z_b}{2}-\zeta\right). \tag{8}$$

Now if we delete the quadratic factors in z_a and z_b —which actually correspond to an edge excitation—we arrive back at (7).

Based on the adiabatic genesis of the pairing $\nu=\frac{1}{2}$ state out of the mother BCS pairing state, and on the strong-binding limit, we expect that half fluxons are anyons with statistical parameter $\theta=\pi/8$. (On the other hand, Moore and Read, using sophisticated techniques of conformal field theory, argued that the charged quasiparticle excitations above the $\nu=\frac{1}{2}$ state they considered—also a form of half fluxons—carry *non-Abelian* statistics. While we believe that Abelian statistics is the generic case, it is conceivable that they are describing a special universality class with extra symmetry.)

Tao and Wu[10] have offered a strong *a priori* argument *against* the possibility of even-denominator Hall states. Specialized to the $\nu=\frac{1}{2}$ case, their argument runs as follows. Charged quasiparticles should be created by adiabatic flux insertion, and at $\nu=\frac{1}{2}$ are expected to be charge-$e/2$ objects with statistical parameter $\theta=\pi/2$. (Halving the flux does not affect the essence of the argument.) These objects are "half fermions," but it is notorious that a composite formed from two half fermions is not a fermion but rather a boson. Thus there is a mismatch between the quantum statistics of the electron and that of the quasiparticles it is supposed to decay into. For odd denominators, $\nu=\frac{1}{3}$ say, this problem does not arise: Three $\frac{1}{3}$ fermions together make a *fermion.*

Given the existence of an incompressible $\nu=\frac{1}{2}$ state, we can run this argument backwards to infer the existence of new sorts of excitations. By inserting two flux tubes *together with an electron* we create a neutral fermion. Such excitations descend, in the evolution through quantum statistics, from the *pair-breaking* excitations of the mother BCS pairing state.

A concrete wave function that implements this construction, describing two localized neutral fermions, is

$$\Psi_{\text{pinned fermions}}=\mathcal{A}(u_a^*u_1+u_a^*v_1)^{2N-3}(u_b^*u_2+v_b^*v_2)^{2N-3}\times\prod_{i=3}^{N}(u_av_i-v_au_i)^2(u_bv_i-v_bu_i)^2\psi_{1/2}[(u_3,v_3),\ldots,(u_N,v_N)]\,. \tag{9}$$

The powers of the factors $u_a^*u_1+v_a^*v_1$ and $u_b^*u_2+v_b^*v_2$ are fixed by homogeneity. The total power is equal to $2N-3$ for all particle coordinates, just as in the ground state.

From the form of the wave function (9), one can infer that the neutral fermions carry spin—that is, orbital angular momentum in the plane—$\frac{1}{2}$. Since spatial rotations simply have the effect of multiplying the spinor coordinates by a phase, the orbital wave functions of a single neutral fermion carry half-odd-integer angular momentum.

Because they are neutral, the pair-breaking excitations are mobile even in the presence of a magnetic field. The energy eigenstates in the neutral fermions will not be localized, and wave functions such as (9) will not represent approximate energy eigenstates. A better trial wave function for energy eigenstates superposes localized states like (9) with different centers, to approximate a momentum eigenstate. Based on arguments similar to those used by Feynman and Bijl for rotons in He II, or on the properties of pair breakers in BCS theory, we expect their dispersion relation will have a dip at a finite momentum. It is remarkably that these two vastly different viewpoints suggest the same qualitative picture.

The presence of separated charged quasiparticles is expected to lower the energy for pair breaking, because each member of the broken pair can separately take advantage of attractive potential provided by the quasiparticles. Indeed in the presence of quasiparticles one can suggest much simpler broken-pair trial wave functions than without them:

$$\Psi_{z_a,z_b}^{\text{broken}}=\mathcal{A}\frac{1}{u_1v_a^*+v_1u_a^*}\frac{1}{u_2v_b^*+v_2u_b^*}\prod_{\substack{i\geq 3\\ i\ \text{odd}}}\frac{1}{u_iv_{i+1}-v_iu_{i+1}}\prod_j(u_jv_a^*+v_ju_a^*)(u_jv_b^*+v_ju_b^*)\psi_{1/2}\,. \tag{10}$$

A generalization of the considerations above, shows that when $2r$ quasiholes are present up to r broken pairs can be accommodated naturally.

Numerical experiments.— We have performed extensive numerical experiments to test the ideas described above. Our results are based on exact diagonalization of the Hamiltonian for small systems on a sphere.[11] The two-body electron interaction is given by a pseudopotential V_l, for the energy of two electrons with total angular momentum $J=2S-l$ ($S=N_\Phi/2$ is the angular momentum of the single-electron states). We calculated the energy spectrum for various systems near filling fraction $\nu=\frac{1}{2}$, for various ratios of V_1/V_3 (all other $V_l=0$), and also in the neighborhood of Coulomb interactions. Some representative results are shown in Figs. 1 and 2.

The major conclusions we infer from our simulations are as follows.

(1) For the potentials containing two partial waves, the lowest-energy state for $N_\phi=2N-3$ and even N occurs at zero angular momentum if $-0.5<V_1/V_3<3$. It is separated by a clear gap from the higher-energy states. These features are all consistent with the existence of a homogeneous incompressible paired Hall state (Fig. 1). However, in case of the Coulomb interaction, there exist several nearly degenerate ground states at different angular momenta. This suggests that the electrons form a compressible charge-density-wave state or a liquid state with strong CDW correlation. If the repulsive V_1 partial wave of the Coulomb repulsion is reduced by, say 18%, a homogeneous incompressible ground state reemerges with a gap $\Delta=0.025e^2/4\pi\epsilon l_0$

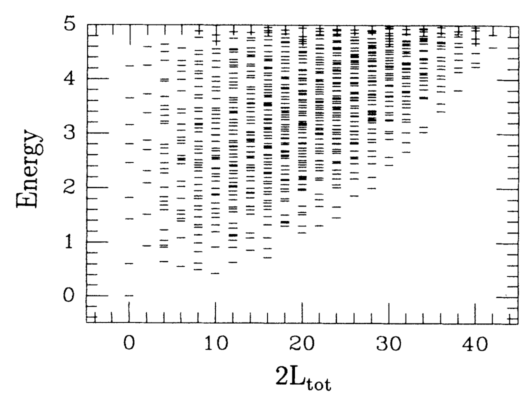

FIG. 1. The energy spectrum of ten electrons on a sphere with seventeen flux quanta. We have chosen $V_1=2$ and $V_3=0.85$. L_{tot} is the total angular momentum of the states.

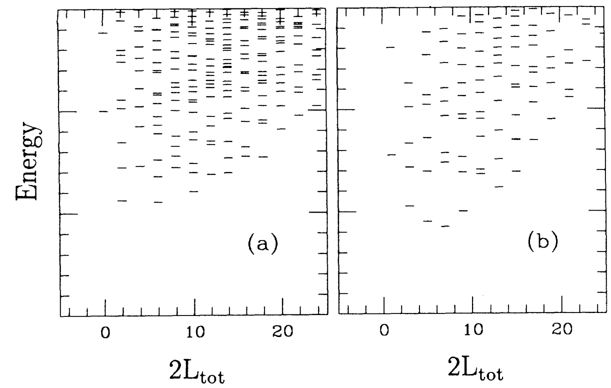

FIG. 2. Energy spectrum of (a) ten electrons and eighteen flux quanta and (b) nine electrons and fifteen flux quanta. The electron interaction is the same as in Fig. 1. The neutral-fermion dispersion curve can be seen clearly in (b).

(for $N=10$).

(2) When N_ϕ differs by 1 from the preferred value, the lowest-energy branch of the spectrum has an alternate even-odd character, and a minimum at small angular momentum. This is consistent with the spectrum for *two identical particles* (the quantum statistics then spaces the allowed angular momenta) with repulsive interactions (on the sphere, these are minimized at small total angular momentum). It matches our expectation that the flux unit is halved [Fig. 2(a)].

(3) When we remove (or add) one electron and two units of flux from the pairing ground state, the low-energy branch has a minimum away from the zero angular momentum [Fig. 2(b)], as we anticipated for the neutral-fermion excitations.

(4) The short-range part of the potential can be dialed toward attraction, without closing the gap. This supports the view that the paired Hall state and Halperin's effective boson state are in the same universality class.

If a pairing Hall state does exist at $\nu=\frac{1}{2}$, how has it managed to escape detection? One possibility is that here as in other cases of superconductivity outside the *s* wave, the pairing correlations are easily destroyed by impurities. Along these lines, it is perhaps suggestive that Tsui and collaborators have seen anomalous features in Hall-effect measurements on ultrahigh mobility samples near $\nu=\frac{1}{2}$.[12]

A longer account of this work is in preparation.[13]

F.W. was supported in part by DOE Grant No. DE-EG02-90ER40542.

[(a)]On leave from Department of Physics, Harvard University, Cambridge, MA 02138.

[1]M. Greiter and F. Wilczek, Mod. Phys. Lett. B **4**, 1063 (1990). See also the discussion in F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990), pp. 80–101.

[2]G. Canright and S. Girvin, Int. J. Mod. Phys. B **3**, 1943 (1989). See especially footnote 30.

[3]B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983); R. Tao, Phys. Rev. B **33**, 2937 (1986).

[4]G. Moore and N. Read, "Non-abelions in the Quantum Hall Effect," Yale University report (unpublished).

[5]For the definition and some basic identities see, for instance, B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard Univ. Press, Cambridge, MA, 1973).

[6]F. Dyson, quoted in J. R. Schrieffer, *Theory of Superconductivity* (Benjamin-Cummings, Reading, MA, 1983), p. 42.

[7]S. Girvin, A. MacDonald, M. P. A. Fisher, S.-J. Rey, and J. Sethna, Phys. Rev. Lett. **65**, 1671 (1990).

[8]M. Greiter and F. Wilczek, "Exact Solutions and the Adiabatic Heuristic for Quantum Hall States" (to be published).

[9]F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[10]R. Tao and Y.-S. Wu, Phys. Rev. B **31**, 1191 (1985).

[11]Related calculations have been performed by G. Fano, F. Ortolani, and E. Tosatti, Nuovo Cimento **9D**, 1337 (1987), for Coulomb interactions at $N_\phi=2N$. No incompressible state was found.

[12]H. Jiang *et al.*, Phys. Rev. B **40**, 12013 (1989). See also R. L. Willet *et al.*, Phys. Rev. Lett. **65**, 112 (1990).

[13]M. Greiter, X.-G. Wen, and F. Wilczek (to be published).