Scaling Solution for Cosmological σ Models at Large N

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We study the evolution of the nonlinear O(N) σ model in an expanding universe in the large-N approximation, where one can solve *exactly* for the scaling field distribution, incorporating the full nonlinear dynamics. This provides a valuable analytic approximation for theories of cosmic global monopoles ($N=3$), global texture ($N=4$), and "nontopological texture" ($N>4$). The analytic results are compared with those from numerical simulations, with good agreement. The probability distribution for mass fluctuations in the linear regime is positively skewed, providing a distinctive test for the theory.

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The problem of the origin of structure in the Universe remains an intriguing and unsolved one. One set of theories is based on a simple physical idea: that the Universe began in a hot, homogeneous state, which as it cooled underwent a symmetry-breaking phase transition.[1] The resulting disordered field distribution, with or without topological defects, enters a scaling solution, in which it orders itself continually on the horizon scale. In the process it generates density perturbations of constant amplitude on each scale at horizon crossing at all times. The earliest example of such a theory was that of cosmic strings,[2] but more recently it was realized that the same mechanism also operates whenever a non-Abelian global symmetry is broken in the early Universe, generically producing global texture.[3] Family symmetry schemes[4] and a class of grand unified theories (GUT's) invented to solve the strong CP problem[5] provide concrete examples of theories where this actually occurs. We have recently devoted some effort to investigating this mechanism in detail: through numerical simulations of texture evolution,[6] calculations of the early formation of galaxies and quasars,[7] large-scale simulations of the large-scale structure formed by texture,[8] and calculations of the anisotropy pattern produced by texture on the microwave sky.[9]

A simple set of theories with broken global symmetries is provided by the "N-vector" models, where O(N) is broken to O($N-1$) by an N-component real vector field. For low N, these theories have topological defects. For $N=1$ one has domain walls, $N=2$ global strings,[2] $N=3$ global monopoles,[10] and $N=4$ global texture, an unstable topological defect.[3] At larger N, there are no topological defects, and the dynamics is simply that of nonlinearly coupled Goldstone boson modes ("nontopological texture"). For all N larger than 1, these theories are potentially interesting theories for the origin of cosmic structure. After the phase transition, at the GUT scale for the theories of interest, the field $\boldsymbol{\phi}$ is closely confined in most of space to the vacuum manifold, $\boldsymbol{\phi}^2=\phi_0^2$, where $\phi_0 \approx m_{\rm GUT}$. However, $\boldsymbol{\phi}$ is free to wander around the vacuum manifold, and these "angular" variations in $\boldsymbol{\phi}$ (the Goldstone modes) lead to an energy density which scales with that of the Universe: $\rho \approx (\partial_i \boldsymbol{\phi})^2 \approx \phi_0^2/t^2$. For $N>2$, the dynamics of the Goldstone modes associated with the vacuum degeneracy is accurately described[3] by the nonlinear sigma model (NLSM).[11] For $N=1$ or 2, most of the energy is localized in the defects (domain walls or global strings), where $\boldsymbol{\phi}$ departs from the vacuum manifold, and the NLSM is not likely to be an accurate description of the ordering dynamics.

Parallel to the work in cosmology on the generation of large-scale structure from the ordering dynamics of nonlinear fields, considerable interest has developed in condensed matter physics in the phenomenon of "self-organized criticality": the ordering dynamics of systems suddenly quenched below the critical temperature.[12] One of the principal analytic tools used here has been the large-N approximation, developed by Mazenko and Zannetti,[13] and Coniglio and Zannetti.[14] In particular, damped spin systems with a nonconserved order parameter which are the nonrelativistic analogs of cosmic texture have been recently investigated both numerically and analytically by Newman, Bray, and Moore.[15,16]

In an expanding universe, the evolution of the N-component field $\boldsymbol{\phi}$ in the NLSM approximation obeys[6]

$$\ddot{\boldsymbol{\phi}}\frac{\alpha(\eta)}{\eta}\dot{\boldsymbol{\phi}}-\nabla^2\boldsymbol{\phi}=-\frac{\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}}{\phi_0^2}\boldsymbol{\phi}\equiv T(\eta,\mathbf{x})\boldsymbol{\phi}\,, \qquad (1)$$

where η is conformal time, $\alpha=2\{d\ln[a(\eta)]/d\ln\eta\}$, and $a(\eta)$ is the expansion factor. T is proportional to the trace of the stress energy tensor for $\boldsymbol{\phi}$. In a flat Friedmann universe, $a(\eta)=\eta/\eta_*+\frac{1}{4}(\eta/\eta_*)^2$, where $\eta_*=(8\pi \times G\rho_{m,\rm eq}/3)^{-1/2}$ defines the transition from radiation to matter domination, and $\rho_{m,\rm eq}$ is the density of matter at equal matter/radiation density. α is a slowly varying function which changes from 2 in the radiation era to 4 in the matter era.

We wish to solve (1) with initial conditions corresponding to a physically reasonable $\boldsymbol{\phi}$ field distribution at some early time η_0, immediately after the symmetry-breaking phase transition. We assume that we can represent $\boldsymbol{\phi}/\phi_0$ as a unit vector of random orientation in each initial correlation volume. This means that for the long-wavelength modes of interest, each Fourier mode of $\boldsymbol{\phi}$ is Gaussian distributed, with a white-noise power spectrum. In addition, we must specify the initial velocities $\dot{\boldsymbol{\phi}}(\eta_0)$. The simplest assumption is that these too have a white-noise power spectrum, although since initial velocities redshift away rapidly, our results will hold for any reasonable initial values of $\dot{\boldsymbol{\phi}}(\eta_0)$.

In the large-N approximation, one simply replaces $T(\mathbf{x},\eta)$ with its spatial average, $\overline{T}(\eta)$. This may be understood as follows. For any given $\overline{T}(\eta)$, (1) is linear, so if $\boldsymbol{\phi}$ begins Gaussian distributed, it remains so for all times. With a Gaussian distribution for N degrees of freedom, the probability distribution for $\boldsymbol{\phi}^2$ becomes more and more sharply peaked about ϕ_0^2 at large N. In particular, the fluctuations in any quadratic quantity like $\boldsymbol{\phi}^2$ or T are proportional to $1/N$, and may be consistently ignored. But ignoring fluctuations amounts to replacing $T(\mathbf{x},\eta)$ by $\overline{T}(\eta)$, so the whole scheme is self-consistent. To find the solution, one finds the general solution for $\boldsymbol{\phi}$ for any $\overline{T}(\eta)$, calculates $\overline{T}(\eta)$ from its definition, and obtains a self-consistency relations, which is then solved for $\overline{T}(\eta)$.

In our case we can actually guess the form of the solution. If α is constant, then from dimensional considerations the only time or length scale is η, so we expect $\overline{T}(\eta)=T_0\phi_0^2/\eta^2$, with T_0 a constant. Furthermore, rather than calculating $\overline{T}(\eta)$ one may simply use the condition $\overline{\langle\phi^2\rangle}(\eta)=\phi_0^2$. This actually guarantees that the calculated value of $\overline{T}(\eta)$ equals $T_0\phi_0^2/\eta^2$, as may be seen from (1) by using an integration by parts, and

$$\overline{\langle\boldsymbol{\phi}^2\rangle}=\phi_0^2\Longrightarrow\overline{\langle\boldsymbol{\phi}\cdot\dot{\boldsymbol{\phi}}\rangle}=0\Longrightarrow\overline{\langle\ddot{\boldsymbol{\phi}}\cdot\boldsymbol{\phi}\rangle}=-\overline{\langle\dot{\boldsymbol{\phi}}^2\rangle}\,.$$

We now solve (1) with T replaced by $T_0\phi_0^2/\eta^2$ in terms of its Fourier modes (treating α as constant):

$$\boldsymbol{\phi}(\eta,\mathbf{x})=\sum_k\boldsymbol{\phi}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\,,$$
$$\boldsymbol{\phi}_{\mathbf{k}}(\eta)=\left(\frac{\eta}{\eta_0}\right)^{(1-\alpha)/2}\frac{J_\nu(k\eta)}{(k\eta_0)^\nu}\boldsymbol{\phi}_{\mathbf{k}}(\eta_0)\,,\qquad(2)$$
$$\nu^2=T_0+\tfrac{1}{4}(1-\alpha)^2\,,$$

with η_0 the initial time. Here we have used $\dot{\boldsymbol{\phi}}_{\mathbf{k}}(\eta_0)=0$, setting the coefficient of the linearly independent Y_ν mode to be vanishingly small for the long-wavelength modes ($k\eta_0\ll1$) of interest. Choosing any reasonable initial magnitude for the initial velocity power spectrum yields the same result.

Now the self-consistency condition reads

$$\overline{\langle\boldsymbol{\phi}^2\rangle}(\eta)=\sum_k\langle\boldsymbol{\phi}_{\mathbf{k}}(\eta)\boldsymbol{\phi}_{-\mathbf{k}}(\eta)\rangle$$
$$=V\int\frac{d^3k}{(2\pi)^3}\langle\boldsymbol{\phi}_{\mathbf{k}}\cdot\boldsymbol{\phi}_{-\mathbf{k}}(\eta_0)\rangle\left(\frac{\eta}{\eta_0}\right)^{(1-\alpha)}\frac{J_\nu^2(k\eta)}{(k\eta_0)^{2\nu}}$$
$$=\phi_0^2\,,\qquad(3)$$

for all time. Taking the initial power spectrum to be white noise, we can rewrite equation (3) as the product of a dimensionless integral of $k\eta$, which we assume converges, times $\eta^{(2\nu-\alpha-2)}$. For the left-hand side to be time independent, we require

$$\nu=1+\alpha/2\,.\qquad(4)$$

But from (2) we now have

$$T_0=\tfrac{3}{4}+\tfrac{3}{2}\alpha\qquad(5)$$

giving $T_0=3.75$ in the radiation era and $T_0=6.75$ in the matter era. We may now check that the integral in Eq. (3) does indeed converge, and note that it is dominated by horizon-scale modes at all times ($k\eta\approx1$). This tells us that the nonlinearity is mainly in horizon-wavelength modes: Modes well inside the horizon are just linear fluctuations. We expect the scaling solution (5) to be stable, and that small deviations of $\overline{T}(\eta)$ relax to zero in a few expansion times.

We now compute the density and pressure of the scalar field from $\overline{T}(\eta)$. For scalar fields, we have

$$\rho=\overline{[\tfrac{1}{2}\dot{\boldsymbol{\phi}}^2+\tfrac{1}{2}(\nabla\cdot\boldsymbol{\phi})^2]}a^{-2},\quad P=\overline{[\tfrac{1}{2}\dot{\boldsymbol{\phi}}^2-\tfrac{1}{6}(\nabla\cdot\boldsymbol{\phi})^2]}a^{-2},$$

and thus $\overline{T}=(\rho-3P)a^2$. We can obtain ρ and P separately by using stress energy conservation (which the solution respects):

$$(\rho\dot{a}^4)=\frac{\dot{a}}{a}(\rho-3P)a^4=\frac{\dot{a}}{a}\overline{T}a^2\,.\qquad(6)$$

Integrating Eq. (6) in the matter era yields $\rho=6.75/a^2\eta^2$, but in the radiation era we find $\rho=3.75\ln(\eta/\eta_0)/a^2\eta^2$, a reflection of the buildup of linear Goldstone modes inside the horizon.[6] Note that at large N, the mean density and pressure are independent of N. We have compared these predictions with the results of numerical simulations described in Ref. 6, and find agreement to within approximately 20% for N from 4 to 10.[17]

For the formation of structure, it is more important to calculate the fluctuations in the density and pressure, since these are responsible for perturbing the matter distribution. In the large-N approximation, all quantities are specified from the two-point correlation function,

$$\langle\phi_{\mathbf{k}}^a(\eta)\phi_{\mathbf{k}'}^b(\eta')\rangle=\phi_0^2\frac{\delta^{ab}}{N}\delta_{\mathbf{k}+\mathbf{k}',0}\Phi_k(\eta)\Phi_k(\eta')\,,$$
$$\Phi_k(\eta)=\frac{\eta^{3/2}}{(k\eta)^\nu}\frac{J_\nu(k\eta)}{\mathcal{N}^{1/2}}\,,\qquad(7)$$
$$\mathcal{N}=\frac{V}{(2\pi)^3}\int d^3xJ_\nu^2(x)x^{-2\nu}\,,$$

where ν is given in (4).

In particular we obtain the power spectrum

$$P(k,\eta)=\langle\phi_{\mathbf{k}}^a(\eta)\phi_{-\mathbf{k}}^a(\eta)\rangle=\frac{\phi_0^2}{\mathcal{N}}\eta^3\frac{J_\nu^2(k\eta)}{(k\eta)^{2\nu}}\,.\qquad(8)$$

We have compared this prediction with numerical simulations of the evolution of the $\boldsymbol{\phi}$ field. Starting with

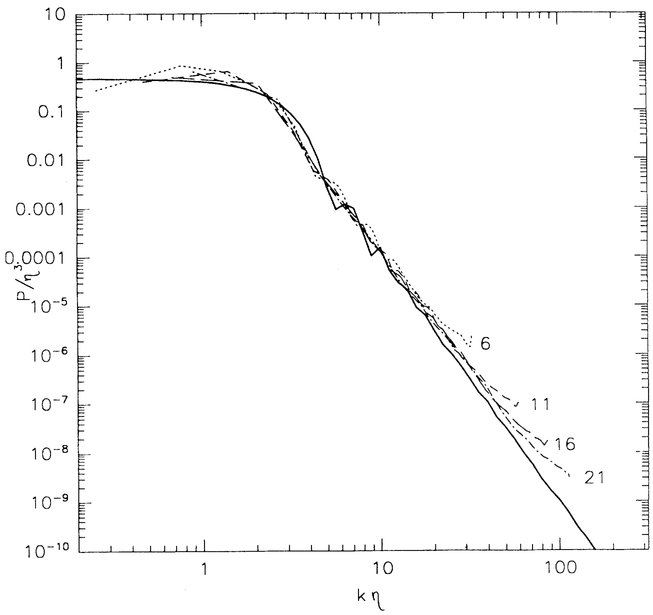

FIG. 1. The power spectrum of the scalar field distribution calculated in a numerical simulation is compared to the analytic prediction, Eq. (8). In the numerical simulation, we evolve an $N=10$ model in a radiation-dominated Universe in a 60^3 box using the techniques described in Refs. 6 and 18. At five different times, we compute the power spectrum convolved with a 3D Parzen window. The labels identify the conformal time η in grid units (Ref. 6). Since we are plotting the spectrum at different times, different snapshots span different ranges of $k\eta$. The heavy solid line shows the theoretical spectrum convolved with the appropriate window function. There are no adjustable parameters in either the theoretical model or in the simulation.

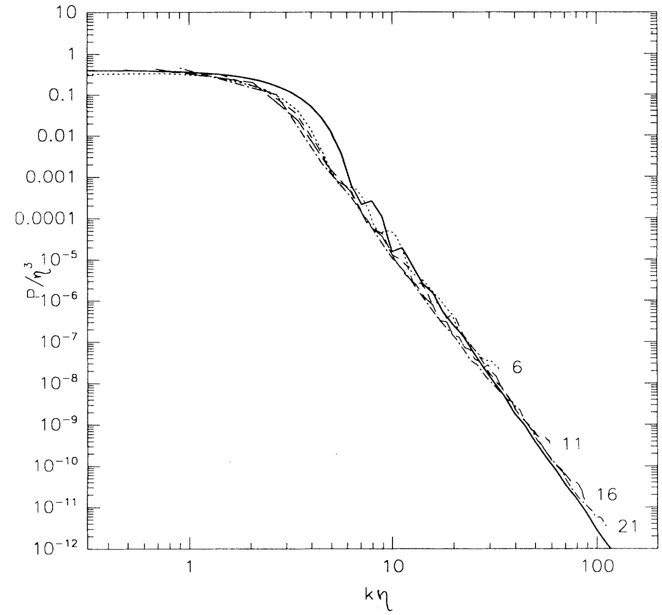

FIG. 2. The same as Fig. 1, but in a matter-dominated Universe.

white-noise initial conditions for the field, we evolve Eq. (1) using the numerical techniques described in Refs. 6 and 18. In Figs. 1 and 2, we show the theoretical power spectra and numerical results for the radiation and matter eras. The simulations scale beautifully, and agree well with the analytic predictions. This increases our confidence in both the numerical simulations and in the large-N approximation. We have repeated the comparison for smaller values of N and found that the agreement is still good down to $N=4$ during both the matter and radiation epochs.

Apart from checking simulations, the scaling solution is useful for calculating quantities of direct cosmological significance: the power spectrum for density fluctuations induced in the matter, the microwave and gravity wave backgrounds, and so on. The calculation of the matter perturbations requires that we extend the scaling solution through the radiation-matter transition. A fairly accurate "adiabatic" solution is obtained by simply replacing the constant α in (4) and (7) by $2d\ln a/d\ln\eta$ throughout. As is seen from (7) this ensures that $\langle\boldsymbol{\phi}^2\rangle = \phi_0^2$ at all times, but the mode functions $\boldsymbol{\phi}_k(\eta)$ do not satisfy the equation of motion exactly. The neglected terms are, however, of order $d\ln a/d\ln\eta$, which has a maximal value of $1/(1+\sqrt{2})^2 \approx 1/6$ near the transition, so it is a reasonable approximation to neglect them.

The source for perturbations in nonrelativistic matter is $S(\mathbf{x}) = 8\pi G\dot{\boldsymbol{\phi}}^2(\mathbf{x})a^{-2}$,[8] and the coefficient of the growing mode in the linear matter perturbations is given by $(\delta\rho/\rho)^g_{\mathbf{k}} = \int d\eta' G_k(\eta')S_{\mathbf{k}}(\eta')$ with $G_k(\eta)$ the Green's function for the mode k. For cold dark matter, G is independent of k, and is given in Ref. 8. For hot dark matter it depends on k, and must be calculated numerically.[19] We find, for $k,l\neq 0$,

$$\left\langle \left(\frac{\delta\rho}{\rho}\right)^g_{\mathbf{k}} \left(\frac{\delta\rho}{\rho}\right)^g_{l} \right\rangle = \frac{8}{N}(4\pi G\phi_0^2)^2 \frac{V\delta_{\mathbf{k}+l,0}}{\mathcal{N}^2}\int \frac{d^3k'}{(2\pi)^3} I_{0,\mathbf{k}}(\mathbf{k}')^2 , \tag{9}$$

where we define

$$I_{l,\mathbf{k}}(\mathbf{k}') = \int d\eta' G_{l+\mathbf{k}}(\eta')\dot{\Phi}_{\mathbf{k}'+l}(\eta')\dot{\Phi}_{\mathbf{k}'-\mathbf{k}}(\eta') ,$$

and similarly for $k,l,m\neq 0$,

$$\left\langle \left(\frac{\delta\rho}{\rho}\right)^g_{\mathbf{k}} \left(\frac{\delta\rho}{\rho}\right)^g_{l} \left(\frac{\delta\rho}{\rho}\right)^g_{\mathbf{m}} \right\rangle = \frac{64}{N^2}(4\pi G\phi_0^2)^3 \frac{V\delta_{\mathbf{k}+l+\mathbf{m},0}}{\mathcal{N}^3}\int \frac{d^3k'}{(2\pi)^3} I_{l,\mathbf{k}}(\mathbf{k}')I_{l,0}(\mathbf{k}')I_{0,\mathbf{k}}(\mathbf{k}') . \tag{10}$$

The mass fluctuation $\delta M/M$ on a given scale is directly related to $(\delta\rho/\rho)_{\mathbf{k}}$, and we see that the rms value of $\delta M/M$ scales as $G\phi_0^2/\sqrt{N}$. Thus for larger N, one requires a larger value of ϕ_0 to produce the same mass fluctuation. The skewness $\langle(\delta M/M)^3\rangle/\langle(\delta M/M)^2\rangle^{3/2} \propto N^{-1/2}$, and the kurtosis $\langle(\delta M/M)^4\rangle/\langle(\delta M/M)^2\rangle^2 \propto 3(1+4/N)$, both approaching the Gaussian value at large N. The power spectrum and skewness may be obtained numerically from (9) and (10), which will provide a valuable complement to the direct numerical evolution of texture done so far.[17] The analytic solution also enables one to calculate the microwave anisotropy and gravity wave backgrounds produced in these theories, ignoring the direct effect of the defects for $N=3$ or 4.[9] It should also be possible to calculate the leading large-N corrections to the quantities discussed above as was done in Ref. 16. It would be interesting to extend the methods described here to the case where cosmic texture is produced during a period of inflation.

The generation of structure through the nonlinear ordering dynamics of Goldstone boson fields provides a promising alternative to the popular Gaussian theories of density fluctuations. The feature of positive skewness in the probability distribution is particularly interesting in the light of the recent detection of positive skewness in the galaxy counts in cells in recent very-large-scale surveys.[20] To end on an optimistic note, with more detailed calculations along the lines reported here, one might be able to determine N from the observations.

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[1]T. W. B. Kibble, J. Phys. A **9**, 1387 (1976).

[2]A. Vilenkin, Phys. Rep. **121**, 263 (1985).

[3]N. Turok, Phys. Rev. Lett. **63**, 2625 (1989).

[4]F. Wilczek, Phys. Rev. Lett. **49**, 1549 (1982); D. Reiss, Phys. Lett. **115B**, 217 (1982); G. Gelmini, S. Nussinov, and T. Yanagida, Nucl. Phys. **B219**, 31 (1983); D. Chang, P. Pal, and G. Senjanovic, Phys. Lett. **153B**, 407 (1985).

[5]A. Nelson, Phys. Lett. **136B**, 387 (1984); S. Barr, Phys. Rev. Lett. **53**, 329 (1984).

[6]D. N. Spergel, N. Turok, W. H. Press, and B. S. Ryden, Phys. Rev. D **43**, 1038 (1991). Because of an inadvertent error in interpreting the numerical results, the scalar field density and pressure given there are both a factor of 2 smaller than they should be.

[7]A. Gooding, D. N. Spergel, and N. Turok, Princeton University Report No. PUP-TH-1207, 1990 [Astrophys. J. Lett. (to be published)].

[8]C. Park, D. N. Spergel, and N. Turok, Princeton University Report No. PUP-TH-1223, 1990 [Astrophys. J. Lett. (to be published)].

[9]N. Turok and D. N. Spergel, Phys. Rev. Lett. **64**, 2736 (1990).

[10]A. Linde, *Particle Physics and Inflationary Cosmology* (Gordon and Breach, New York, 1989); Barriol and A. Vilenkin, Phys. Rev. Lett. **63**, 341 (1989); D. Bennett and S. Rhie, Phys. Rev. Lett. **65**, 1709 (1990).

[11]See, e.g., R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[12]For reviews, see J. D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 8, p. 287; H. K. Furukawa, Adv. Phys. **34**, 703 (1985); K. Binder, Rep. Prog. Phys. **50**, 783 (1987).

[13]G. Mazenko and M. Zannetti, Phys. Rev. B **32**, 4565 (1985).

[14]A. Coniglio and M. Zannetti, Europhys. Lett. **10**, 575 (1989).

[15]T. J. Newman, A. J. Bray, and M. A. Moore, Phys. Rev. B **42**, 4514 (1990); see also A. J. Bray, Phys. Rev. B **41**, 6724 (1990); Phys. Rev. Lett. **62**, 2841 (1989).

[16]T. J. Newman and A. J. Bray, J. Phys. A **23**, L279 (1990).

[17]D. Spergel and N. Turok (to be published).

[18]W. H. Press, B. Ryden and D. N. Spergel, Astrophys. J. **347**, 590 (1989).

[19]R. Cen, J. P. Ostriker, D. N. Spergel, and N. Turok, Princeton University Report No. PUPT-1255-91 (to be published).

[20]G. Efstathiou, N. Kaiser, W. Saunders, A. Lawrence, M. Rowan-Robinson, R. S. Ellis, and C. S. Frenk, Mon. Not. Roy. Astron. Soc. **247**, 10P (1991). Note that nonlinear evolution of a Gaussian field can also produce skewness in the mass distribution; see P. J. E. Peebles, *The Large-Scale Structure of the Universe* (Princeton Univ. Press, Princeton, NJ, 1980), p. 74.