PHYSICAL REVIEW LETTERS

VOLUME 66 | 17 JUNE 1991 | NUMBER 24

Nonequilibrium Potentials for Dynamical Systems with Fractal Attractors or Repellers

R. Graham and A. Hamm

Fachbereich Physik, Universität Gesamthochschule D4300 Essen 1, Germany

T. Tél[(a)]

Institut für Festkörperforschung, Forschungszentrum, D5170 Jülich, Germany

(Received 7 September 1990)

The concept of a nonequilibrium potential is applied to dynamical systems with fractal attractors or repellers. In particular, we study the case of the Feigenbaum attractor in one-dimensional maps and the case of hyperbolic strange attractors or repellers in two-dimensional maps. The potential-height distribution of the latter is shown to exhibit multifractal features; that of the Feigenbaum attractor is characterized by a single number, the universal exponent for noisy period-doubling bifurcations.

PACS numbers: 05.45.+b, 02.50.+s, 05.40.+j, 05.70.Ln

The concept of a nonequilibrium potential for a dynamical system arises in two different ways.

(i) Let a dynamical system be described by a set of differential equations $\dot{q}^\nu = K^\nu(q)$ and let some positive-definite "transport matrix" $Q^{\nu\mu}(q)$ be given. A nonequilibrium potential $\Phi(q)$ may be introduced as a continuous but not necessarily continuously differentiable solution of $K^\nu(q) = r^\nu(q) - \frac{1}{2} Q^{\nu\mu}(q)\partial_\mu\Phi(q)$ and $r^\nu(q)\partial_\nu\Phi(q) = 0$ with the boundary condition that Φ should be minimal in attractors. The above conditions combined determine nonequilibrium potentials locally as solutions of a Hamilton-Jacobi equation, $K^\nu\partial_\nu\Phi + \frac{1}{2} Q^{\nu\mu}\partial_\nu\Phi\partial_\mu\Phi = 0$, and Φ decreases like $\dot{\Phi}(q(t)) = -\frac{1}{2} Q^{\nu\mu}\partial_\nu\Phi\partial_\mu\Phi$. The dynamics $\dot{q}^\nu = K^\nu(q)$ may be interpreted as relaxation in the potential Φ subject to the Φ-conserving part $r^\nu(q)$ of the drift $K^\nu(q)$. Φ can equivalently be determined by solving a set of Hamiltonian differential equations or by minimizing the action integral

$$S[q] = \frac{1}{2}\int_0^T d\tau\, Q_{\nu\mu}(q)[\dot{q}^\nu - K^\nu(q)][\dot{q}^\mu - K^\mu(q)]\,, \qquad (1)$$

where $Q_{\nu\mu}$ is the matrix inverse of $Q^{\nu\mu}$. Among different coexisting local minima of the action integral the infimum is chosen.

(ii) Let the same dynamical system be perturbed by Gaussian white noise in the sense of Ito,[1] $\dot{q}^\nu = K^\nu(q) + g^{\nu i}(q)\xi_i(t)$, of infinitesimal intensity $\langle\xi_i(t)\xi_j(0)\rangle = \eta\delta_{ij}(t)$ and with "diffusion" matrix $\sum_i g^{\nu i}(q)g^{\mu i}(q) = Q^{\nu\mu}(q)$, and ask for the probability density $W_\eta(q)$ in steady state, provided it exists, and the mean first exit time $\langle\tau_\eta\rangle$ out of a basin G with boundary ∂G of an attractor A. Asymptotically, for $\eta \to 0$,

$$W_\eta(q) \sim \exp[-\Phi(q)/\eta]\,, \quad \langle\tau_\eta\rangle \sim \exp[\Delta\Phi(q)/\eta]\,, \qquad (2)$$

where $\Phi(q)$ defined by (2) is also a solution of the Hamiltonian-Jacobi equation with diffusion matrix $Q^{\nu\mu}$, and $\Delta\Phi := \min\{\Phi(y) - \Phi(a) : a \in A, y \in \partial G\}$. Nonequilibrium potentials and first exit times in the weak-noise limit have been computed for many systems both for flows (cf., e.g., Refs. 2–4 and references therein) and for maps.[5–9] A mathematical theory of the asymptotic estimates (2) has been developed by Freidlin and Wentzell[10] for continuous time and transferred to the discrete case in Ref. 11. However, the concept of nonequilibrium potential has not been applied so far to systems with strange invariant sets. This is the purpose of the present paper.

In the following we shall use the definition provided by Eq. (2) but shall employ the action integral (1) to determine Φ. Systems with fractal attractors are most easily investigated by means of return maps. Thus we use Eqs.

© 1991 The American Physical Society

(1) and (2) in their corresponding versions for maps, $q_{n+1}=f(q_n)+\sqrt{\eta Q}\xi_n$. Here f is a map of an interval or of some compact subset of the plane into itself. For simplicity we shall assume Q to be a constant equal to 1. (ξ_n) is a δ-correlated sequence of bounded random variables with truncated Gaussian density[12] in order to render the perturbed noisy map well defined.

In analogy to Eq. (1) the action S of a sequence $(q_j)_{0\le j\le N}$ is defined by

$$S[(q_j)]=\tfrac{1}{2}\sum_{j=0}^{N-1}[q_{i+1}-f(q_i)]^2 .$$

Let $V_x^N(y)$ be its minimal values for fixed N and given initial and final points $q_0=x$, $q_N=y$. In analogy to Hamilton's equations fulfilled by the minimizing path to the action (1) in continuous-time cases, one finds for maps that a minimizing sequence $(q_n)_{0\le n\le N}$ satisfies the equations $q_{n+1}=f(q_n)+p_{n+1}$, $p_{n+1}=p_n/f'(q_n)$, where $f'(q_n)\neq 0$ is assumed, and where $q_0=x$ and p_0 is to be chosen to ensure $q_N=y$ (cf. Refs. 6-9 and 13). The infimal action from x to y is defined by $V_x(y):=\inf\{V_x^N(y):N\ge 1\}$. In systems with only one attractor A the nonequilibrium potential $\Phi(x)$, satisfying the estimates (2) under certain conditions, is $\Phi(x)=V_a(x)$ with $a\in A$. The Hamilton-Jacobi equation for Φ has been derived in Ref. 5.

Let us now look at examples furnished by the family of quadratic maps of the interval, $f_\mu(x)=1-\mu x^2$ with $0<\mu\le 2$. The attractor is then either a periodic orbit, a chaotic interval (strange attractor), or an "attracting" Cantor set. The latter type occurs, for instance, for $\mu=\mu_\infty$, the Feigenbaum accumulation point (see, e.g., Ref. 14). The nonequilibrium potential for this case is shown in Fig. 1. It vanishes on the attracting Cantor set and has maxima $V^{(n)}$ on all unstable periodic orbits of length 2^{n-1} inside the gaps of the Cantor set. The heights of the potential maxima depend only on the length of the unstable periodic orbit and not on the gap size. Traditionally (see Refs. 14 and 15), the universal noise scaling behavior for the Feigenbaum attractor is investigated by linearization in the noise term of the renormalized iterated noisy map, which leads to the detection of the noise scaling constant $\kappa\approx 6.619\ldots$. That procedure allows only indirect conclusions on the observable steady-state distribution. In contrast, the asymptotic scaling properties of the potential are directly connected with those of the steady-state distribution by Eq. (2).[16] The most conspicuous property is the scaling law for the maxima $V^{(n)}$:

$$\lim_{n\to\infty}(1/n)\ln V^{(n)}=-2\ln\kappa . \qquad (3)$$

Heuristically, the relation of (3) to the above-mentioned results on the iterated noisy map may be seen as follows: By the period-doubling property the potential $V^{(n)}$ implies a potential $\tilde V^{(n+1)}\sim\alpha^{-2}V^{(n)}$ in the next gap (the tilde indicates that this potential belongs to the two-times-iterated map, for which the noise amplitude has grown by a factor $\alpha^{-1}\kappa$), and $V^{(n+1)}\sim\alpha^2\kappa^{-2}\tilde V^{(n+1)}\sim\kappa^{-2}V^{(n)}$. A proof of (3), connecting the scaling of the potential maxima with the free energy of the Feigenbaum attractor,[14] will be given in a separate publication.[17] This approach of nonequilibrium potentials, which can also be formulated as a renormalization-group theory of minimizing paths, leads to an alternate and more general derivation of the universal results on noise scaling by uncovering directly the universal properties of the underlying steady-state distribution.

In the periodic windows a repelling Cantor set may coexist with the attracting periodic orbit. On this repeller the potential is constant. The potential must even have the same value on all gaps of the Cantor set, except those containing the attracting periodic orbit: The potential in the gaps of the Cantor set is larger than or equal to its value on the repeller because it can be shown that the action of any sequence from the attractor to points in the gaps not containing the attractor can be lowered by inserting into the sequence points closer to the repeller; on the other hand, the potential in the gaps is smaller than or equal to its value on the repeller, because any point within a gap can be reached via some orbit of the deterministic map starting arbitrarily close to the repeller (namely, in a sufficiently small gap).

Let us turn now to hyperbolic strange sets generated by two-dimensional dissipative maps as strange attractors or repellers.[18] Because of hyperbolicity, stable and unstable directions are defined everywhere. The strange attractor is the closure of the unstable manifolds of some periodic points. Therefore, the potential is constant on these unstable manifolds. A strange repeller is a Cantor set along both the unstable and the stable directions. Nevertheless, as in the case of one-dimensional maps, the nonequilibrium potential is constant along the unstable manifold within the repelling Cantor set. On the other hand, for both strange attractors and repellers, the non-

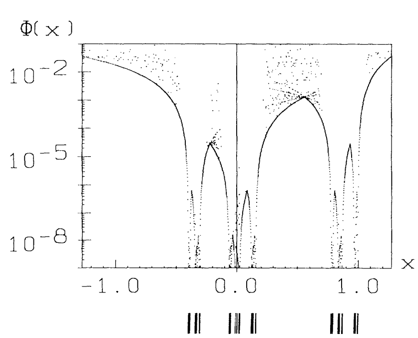

FIG. 1. Nonequilibrium potential of the quadratic map in the Feigenbaum case (attractor marked by vertical dashes), obtained as lower envelope of the action of minimal sequences starting with $q_0=0$ and $p_1=\epsilon$, where ϵ varies from 3×10^{-6} to 0.27.

equilibrium potential is a rapidly varying function in the stable direction and increases locally as one moves along this direction away from the strange set into its gaps. Consider the n-fold ($n \gg 1$) image of some region containing the strange set. It consists of branches lying along the unstable direction which provides an nth approximant to the unstable manifolds, and of gaps between these branches. Let $\Phi_j^{(n)}(s,u)$ be the potential in the jth gap at a distance s along the stable direction from the nearest branch of the unstable manifold and u a coordinate on the latter. The nonequilibrium potential, determined as the infimum over all paths from the attractor to a given point, is, in general, a nonlocal quantity. However, for large n, i.e., in small gaps, the potential increases quadratically with s:

$$\Phi_j^{(n)} = \tfrac{1}{2} G_j^{(n)}(u) s^2 . \tag{4}$$

The coefficient $G_j^{(n)}(u)$ contains the directional derivatives of the map in the stable direction at the points of the minimizing sequence. Let m^2 and M^2 be the minimum and maximum, respectively, of the square of these derivatives. Then, independent of n, j, and u, $1-M^2 \le G_j^{(n)}(u) \le 1-m^2$. It follows that the potential maximum $V_j^{(n)}$ in the gap j at the nth level scales as the square $(l_j^{(n)})^2$ of the size of the gap along the stable direction. This size scales as the width of the branch containing the gap at the previous level $n-1$. It is known from the periodic-orbit theory of chaos (see, e.g., Ref. 19) that the width of branch j at level n can be estimated as the second eigenvalue $\exp(\lambda_{2j} n) < 1$ of the unique unstable n-cycle with a point in this branch. Therefore, $l_j^{(n)} \sim \exp[\lambda_{2j}(n-1)]$, and for $n \to \infty$,

$$V_j^{(n)} \sim (l_j^{(n)})^2 \sim \exp(-2|\lambda_{2j}|n) . \tag{5}$$

As a consequence, the last level n^* which can be completely resolved at noise intensity η is approximately $n^* \sim \ln(1/\eta)/2|\lambda_2|_{\max}$.

Relation (5) implies that the multifractal scaling of the $l_j^{(n)}$ can be transferred to corresponding scalings of the potential maxima $V_j^{(n)}$. For example, for n large, the number $N_c(\lambda_2)$ of n-cycles with a given λ_2 increases exponentially like $N_c(\lambda_2) \sim \exp[n g_2(\lambda_2)]$.[20] Hence, the number $N_g(\gamma)$ of gaps with sizes scaling as $l \sim \exp(-\gamma n)$ and the number $N_v(\gamma)$ of potential maxima V scaling as $V \sim \exp(-2\gamma n)$ both increase as $N_g(\gamma) \sim N_v(\gamma) \sim \exp[n g(\gamma)]$, where $g(\gamma)$ is the multifractal spectrum[21] for the potential-height distribution. Since $N_c(-\gamma) \sim N_v(\gamma)$, we obtain $g(\gamma) = g_2(-\gamma)$, i.e., the height distribution is completely defined by the spectrum of the second local Lyapunov exponent. Conversely, by measuring the potential, one obtains information about the distribution of the second Lyapunov exponents which are difficult to access otherwise.

For maps with constant Jacobian J, the function $g(\gamma)$ can be connected with other observables too. Here we give the result for chaotic attractors only. Writing the

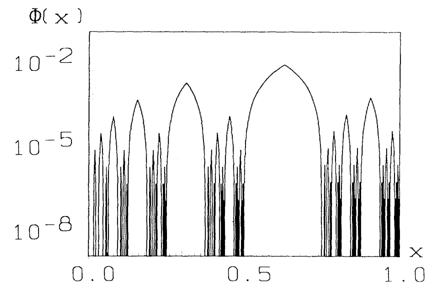

FIG. 2. Cross section through the nonequilibrium potential of the baker's map ($r_1 = \frac{1}{2}$, $r_2 = \frac{1}{4}$) in the x direction.

total generalized dimension as $D_q = 1 + D_q^{(2)}$, the multifractal spectrum[21] $f_2(\alpha_2)$ of the partial dimension $D_q^{(2)}$ can be expressed as $f_2(\alpha_2) = g(\gamma)/\gamma$, with $\alpha_2 = 1 + \ln|J|/\gamma$. The dynamical entropies[20] K_q are found to be $K_q = (H_q + q\ln|J|)/(q-1)$, where H_q denotes the Legendre transform of $g(\gamma)$.

We emphasize the sharp contrast to the Feigenbaum attractor. The latter is not hyperbolic and not even a real attractor because unstable periodic orbits exist even in arbitrarily small gaps. These have a strong influence on the potential maxima which no longer scale with the gap size. Formally, the distribution is a monofractal: g is defined in a single point $\gamma = \ln\kappa$, where $g = \ln 2$.

As an explicit example let us consider the strange attractor generated by the dissipative baker's map:[18,20] $x_{n+1} = r_1 x_n$, $y_{n+1} = s y_n$ if $0 \le y \le \frac{1}{2}$, and $x_{n+1} = 1 - r_2(1-x_n)$, $y_{n+1} = 1 - s(1-y_n)$ if $\frac{1}{2} < y \le 1$, with $r_{1,2}$ positive and $r_1 + r_2 < 1$. The case of a strange repeller appears for $s > 2$. In the following we consider the strange attractor and put $s = 2$.

The gap sizes at the level $n+1$ are $l_{\gamma,i}^{(n+1)} = (1 - r_1 - r_2) r_1^m r_2^{n-m} \sim e^{-\gamma(n+1)}$ with $0 \le m \le n$ and $1 \le i \le N_g(\gamma)$. Here γ is obtained for large n as $\gamma = -x\ln r_1 - (1-x)\ln r_2$, where $x = m/n$, and the number $N_g(\gamma)$ of gaps of a given length determined by x or γ is $N_g(\gamma) = \binom{n}{m} \sim \exp[n g(\gamma)]$.

The nonequilibrium potential of the baker's map along the stable x direction is shown in Fig. 2 for $r_1 = \frac{1}{2}$ and $r_2 = \frac{1}{4}$. Owing to piecewise linearity, Eq. (4) holds exactly for all n. The coefficients $G_j^{(n)}(u)$ are independent of u and can be written down explicitly. The potential maxima for gaps of the same length are not equal, but differ at most by a factor $(1-r_2^2)/(1-r_1^2) = \frac{5}{4}$, which does not influence the scaling behavior. Thus we can conclude $N_v(\gamma) \sim \exp[n g(\gamma)]$.

Finally, in order to exhibit an application of the nonequilibrium potential, let us consider a one-dimensional map of the form shown in the upper part of Fig. 3, where two attractors coexist and are separated by a strange repeller. How can the stability of the two attractors be compared? As discussed above, the nonequilibrium potential is constant on the interval embedding the strange

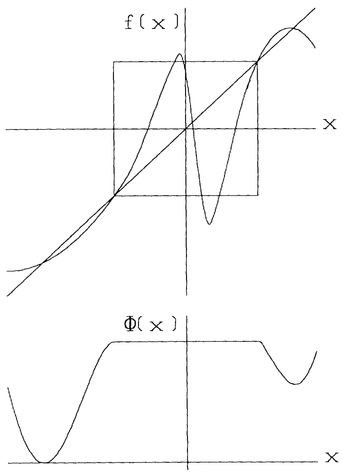

FIG. 3. A function $f(x)$ with two stable fixed points separated by a fractal repeller and its nonequilibrium potential $\Phi(x)$.

repeller as shown in the lower part of Fig. 3. The depths of the potential wells surrounding the two attractors are therefore determined relative to this constant level. In this way the stability of the two attractors against small stochastic perturbations without memory and with constant intensity can be compared with each other in an objective way. Clearly, there are applications for which this measure of stability is more objective than, e.g., the one based on the relative size of the domains of attraction of the coexisting attractors.

We would like to thank P. Talkner and P. Reimann for discussions about their work and for sending us a copy of Ref. 8. Two of us (R.G. and A.H.) would like to thank the Deutsche Forschungsgemeinschaft for support through the Sonderforschungsbereich No. 237.

[(a)]On leave from Eötvös University, Budapest, Hungary.

[1]K. Ito, Proc. Imp. Acad. **20**, 519 (1944); C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[2]R. Graham, in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. E. McClintock (Cambridge Univ. Press, Cambridge, 1989), Vol. 1, p. 225.

[3]P. Haenggi, J. Stat. Phys. **42**, 105 (1987).

[4]R. L. Kautz, Phys. Lett. A **125**, 315 (1987); Phys. Rev. A **38**, 2066 (1988).

[5]P . Talkner, P. Haenggi, E. Freidkin, and D. Trautmann, J. Stat. Phys. **48**, 231 (1987); P. Talkner and P. Hänggi, in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. E. McClintock (Cambridge Univ. Press, Cambridge, 1989), Vol. 2, p. 87.

[6]P. Grassberger, J. Phys. A **22**, 3283 (1989).

[7]P. D. Beale, Phys. Rev. A **40**, 3998 (1989).

[8]P. Reimann, Diplomarbeit, Institut für Theoretische Physik, Basel, 1989 (unpublished).

[9]P. Reimann and P. Talkner, Helv. Phys. Acta **63**, 845 (1990); (to be published).

[10]M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems* (Springer, New York, 1984).

[11]Yu. Kifer, *Random Perturbations of Dynamical Systems* (Birkhäuser, Boston, 1988).

[12]H. Haken and G. Mayer-Kress, Z. Phys. B **43**, 185 (1981).

[13]M. L. Blank, SIAM Theor. Prob. Appl. **33**, 612 (1989).

[14]E. B. Vul, Ya. G. Sinai, and K. M. Khanin, Russ. Math. Surv. **39**, (3) 1 (1984).

[15]J. Crutchfield, M. Nauenberg, and J. Rudnick, Phys. Rev. Lett. **46**, 933 (1981); B. I. Shraiman, C. E. Wayne, and P. C. Martin, Phys. Rev. Lett. **46**, 935 (1981).

[16]There is an interesting similarity between Figs. 1 and 2 of G. Mayer-Kress and H. Haken, J. Stat. Phys. **26**, 149 (1981). For a detailed discussion of the relation of the two pictures revealing a different scaling behavior we refer to A. Hamm and R. Graham (to be published).

[17]Hamm and Graham (Ref. 16).

[18]T. Tél, in *Directions in Chaos,* edited by Hao Bai-lin (World Scientific, Singapore, 1990), Vol. 3, pp. 149–211.

[19]P. Cvitanovic, G. H. Gunaratne, and I. Procaccia, Phys. Rev. A **38**, 1503 (1988).

[20]P. Grassberger, in *Chaos,* edited by A. V. Holden (Manchester Univ. Press, Manchester, 1986); P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988).

[21]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).