Exponential Tails and Random Advection

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A simple phenomenological model for a passive scalar subject to a mean gradient and undergoing random advection plus molecular mixing yields an exponential distribution as a consequence of general analytic properties of the steady-state solution. The result persists in the presence of a coherent mean flow. Experimental realizations of our model are proposed.

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In turbulence theory, the wave-number spectrum has played a predominant role ever since Kolmogorov predicted the exponent; and the cascade ideas underlying that prediction have become the basis for all turbulence phenomenology.[1] The probability distribution function (PDF) seemed less interesting because for the large scales of motion of greatest interest in most engineering problems, most of the fluctuations of velocity and passive contaminants follow a Gaussian distribution.[2] However, proposals for corrections to Kolmogorov's $\frac{5}{3}$ law quickly lead to measurements of fourth and higher moments of the velocity derivatives and van Atta and Chen in 1970 published a PDF with exponential tails for this quantity.[3] Much additional data have been accumulated for velocity and scalar derivatives and differences in the inertial range which display exponential tails.[4,5] Models based on cascade ideas[6] or the saturation of stretching have been proposed[7] which are most plausible for derivative quantities.[8]

Our interest in the pervasiveness of exponential tails stems from the Chicago convection experiments[9] which showed a "universal" exponential in the PDF for the temperature itself (not the derivatives), in the center of the cell. In certain regimes, the exponential only appeared after high-pass filtering. In our view buoyancy effects are not necessary for the exponential tails in these experiments,[10] nor is a multistep cascade.

We model the mixing of a scalar θ by random advection in a velocity field with correlation length ξ, characteristic magnitude v, and correlation time $\tau \sim \xi/v$ in the presence of a mean θ gradient. The Peclet number $v\xi/\kappa$, where κ is the molecular diffusivity, is sufficiently large so that within the confines of a Kolmogorov-cascade picture, the random stretching and folding produced by the gradients of v within a scale ξ may be approximated by an eddy diffusivity $\sim \xi\tau \gg \kappa$. Substantial enhancement of molecular mixing within a correlation volume ξ^d in a time τ results. Simultaneously, there is also a substantial probability that a blob of fluid $\sim \xi^d$ is simply advected as a unit a distance ξ leaving the PDF of θ integrated over ξ^d.

When ξ is smaller than all other lengths, we interpret θ as integrated, or coarse grained, over a volume ξ^d around $\mathbf{r}$ and define a PDF, $P(\theta,\mathbf{r},t)$, by averaging over the random velocity. The mean scalar gradient is maintained by the boundary conditions $P=\delta(\theta)$, $\delta(\theta-1)$, respectively, on the walls $x=0,1$. Then P is only nonzero for $\theta\in[0,1]$. By definition, $\int P\,d\theta=1$ and $\langle\theta\rangle=\int\theta P\,d\theta$ is the scalar density with respect to $\mathbf{r}$ so that $\int\langle\theta\rangle d^3r$ must be conserved. It will be convenient to define the Fourier transform of P with respect to θ, P_k, which serves as a generating function for the moments, $\langle\theta^n\rangle=(-i\,\partial_k)^n P_k|_{k=0}$.

Our model is[11]

$$P_k(\mathbf{r},t+\tau) = -\mathbf{U}(\mathbf{r},t)\cdot\nabla P_k(\mathbf{r},t) + D\nabla^2 P_k(\mathbf{r},t) + P_{k/2}^2(\mathbf{r},t)\,, \qquad (1)$$

where $\mathbf{U}$ represents the translation affected by any large-scale coherent flow that may be present in a time τ. Clearly, (1) respects the norm $P_{k=0}=1$ and conserves $\int\theta\,d^3r$. The "diffusion" of P with $D\sim\xi^2$ arises from the random advection, as one can see by modeling its effect on P_k as a random displacement by $\mathbf{l}$ with $\langle l^2\rangle\sim\xi^2$ and then expanding $P_k=\langle\exp[ik\theta(\mathbf{r}+\mathbf{l},t)]\rangle$ in a Taylor series to $O(l^2)$. It is distinguished by the property that it preserves $\int\langle\theta^n\rangle d^3r$.

The nonlinear term which models the mixing due to the eddy-enhanced diffusivity would appear more familiar as a convolution in θ variables. It results from approximating the mixing during a time τ as the average of θ at points separated by a diffusion length, viz., $\theta(x)\to\frac{1}{2}[\theta(x-\xi/2)+\theta(x+\xi/2)]$. One then exploits the short correlation length of θ, $\sim\xi$, to factor the joint probability of θ at $x\pm\xi/2$ as the product of the single point PDF's. The spacial separation is immaterial for what follows and it should be emphasized that θ inherits

its correlation length and time from v.[12] Our approximation to the mixing reduces connected averages, $\langle\theta^n\rangle_c = (-i\partial_k)^n \ln P_k|_{k=0}$, by 2^{1-n} each iteration and thereby relaxes an arbitrary distribution to a Gaussian one. Note that the mean is invariant and the variance decreases by 2 as follows directly from the average above.

Equation (1) which, schematically at least, results from integrating over an integral time τ is akin to a map and potentially allows for a simpler phenomenology than would differential equations. In particular, the velocity is given time to transport θ through one course-graining cell, i.e., a distance ξ. The mixing assumes a simple form because with large enough Peclet number to justify an eddy diffusivity $\sim v^2\tau \sim \xi^2/\tau$, diffusion within a correlation volume will decrease the variance by ~ 2.

We begin by examining the stationary solutions of (1) with $\mathbf{U}=\mathbf{0}$ in one dimension which for large and small D illustrate two very different probability distributions that share a common mean $\langle\theta\rangle = x$. For $D \gg 1$ we can rewrite (1),

$$P_k(x) = xe^{ik} + (1-x) + \frac{1}{D}\int G(x,x')(P_{k/2}^2 - P_k)(x')dx' , \quad (2)$$

where G is the Green's function which inverts the operator ∂_x^2 with zero boundary conditions. The solution by iteration consists of a sum of δ functions located at $\theta=0$, 1, and $(2p+1)/2^n$, $p < 2^{n-2}n \geq 2$. Note again the interpretation of D as random advection; physical diffusion of the θ field would not preserve the δ functions.

The opposite limit $D \ll 1$ could be achieved in practice by letting the system size increase keeping the other parameters fixed. For x away from the boundaries and since $\partial_x^2\langle\theta\rangle = 0$, one expects P_k to have a fixed shape shifted by the mean, i.e.,

$$P_k = e^{ik\langle\theta\rangle}\chi(k) \quad (3)$$

with this ansatz (1) can be solved exactly to yield

$$\chi^{-1}(k) = (1+Dk^2)\prod_{i=1}^{\infty}(1+Dk^2/2^{2i})^{2^i} . \quad (4)$$

It may be proven that the infinite product in (4) converges absolutely for any complex k, making its inverse χ a meromorphic function with a strip of analyticity around the real axis. Since χ decays rapidly for large real k, there are no δ functions in P and the PDF is continuous. An analyticity strip implies exponential tails which for (3) and (4) appear as

$$P \sim \exp(-|\theta - \langle\theta\rangle|/\sqrt{D}) , \quad (5)$$

with no discernible Gaussian in the center. Note that in our units the exponential in (5) is nondimensionalized by $d\langle\theta\rangle/dx = 1$.

Clearly (3) does not satisfy the boundary conditions, but the required boundary layers can be understood by considering the crossover between the bulk solutions for large and small D. This appears as a well-defined bifurcation for (1) but within a more realistic model there are never true δ functions. Nevertheless, the phenomenon is interesting.

Imagine P has both continuous and "discrete" components and denote the weight of the δ functions, which all have positive coefficients by $W(x)$. Then $W \leq 1$ and $W(0) = W(1) = 1$. The discrete portion of P, P_d, alone must solve (1) since ∇_r^2 and the convolution preserve a δ function in θ. We then derive an equation for W by setting $k=0$ in the P_d equation and find

$$W = D\partial_x^2 W + W^2 . \quad (6)$$

By interpreting (6) as a mechanical system, we see that for large D the only solution is W fixed at 1, the minimum of the potential. Solutions with $W < 1$ become possible for $D < 1/\pi^2$, where $W \equiv 1$ becomes unstable and $W(x)$ is explicitly computable.

For $D \ll 1$, $W \sim e^{-1/\sqrt{D}}$ in the bulk, so there is a small additive correction to (3) which becomes $O(1)$ in the boundary layers. One still expects $P_c \equiv P - P_d$ to be exponentially distributed since if one formally solves $(1 - D\partial_x^2)P_{k,c} = P_{k/2}^2 - P_{k/2,d}^2$ by iteration, the homogeneous solution is zero because $P_c = 0$ at $x = 0,1$. There are then no terms on the right which would make $(1 - D\partial_x^2)^{-1}$ singular as $k \to 0$ and thus we expect P_c to have a strip of analyticity in k. All of the above properties have been confirmed by solving (1) numerically with $\mathbf{U}=0$.

Our treatment of the $D \ll 1$ limit can readily be generalized to include $\mathbf{U} \neq 0$ in which case $\langle\theta\rangle(r)$ is nontrivial and obeys

$$\mathbf{U}\cdot\nabla\langle\theta\rangle = D\nabla^2\langle\theta\rangle . \quad (7)$$

By making the ansatz $P_k(r) = e^{ik\langle\theta\rangle(r)}\chi_k(r)$, a formal iteration can be set up for χ beginning from (4) with $D \to D(\nabla\langle\theta\rangle)^2$ and with additional terms coming solely from the curvature in $\langle\theta\rangle$, i.e., $\nabla^2\langle\theta\rangle$.

An interesting generalization of (1) models the enhanced mixing that results from stretching and folding on the large scales as a one-dimensional map f of the interval $x \in [0,1]$ onto itself that is ergodic has a positive Liapunov exponent and 0,1 as unstable fixed points. To generalize, we replace

$$P_{k/2}^2(x) \to \lambda P_{k/2}^2(x) + (1-\lambda)\prod_i P_{k_i}(f_i^{-1}(x)) , \quad (8)$$

where the product is over all inverse images of x, $k_i = (\partial_x f_i^{-1})k$, and $0 < \lambda < 1$. The last term in (8) results from transformation of a given realization of $\theta(x)$ under the Frobenius-Perron operator

$$\theta'(x) = \sum_i (\partial_x f_i^{-1})\theta(x_i) \quad (9)$$

plus the assumption that $\theta(x_i)$ are independently distributed. Note that (9) properly conserves $\int\theta dx$, and the resemblance to our treatment of diffusion.

Suppose that we add a term $\kappa\partial_x^2\theta$ to (9), impose our boundary conditions, and ask for a steady solution. When f consists of line segments joining the points (0,0), $(\frac{1}{3},1)$, $(\frac{2}{3},0)$, and (1,1) one finds for small κ approximately

$$\theta(x\le 0.5)-0.5=\frac{x-0.5}{1+\text{const}\times x/\sqrt{\kappa}} \qquad (10)$$

and $\theta(x>0.5)=1-\theta(x\le 0.5)$. [The same equation for $\langle\theta\rangle$ would arise from (1) and (8) with $\kappa\to D/(1-\lambda)$; but we emphasize again that the stationary PDF has nonzero variance only in the presence of random advection.[12]] In analogy to a convection experiment, the mean gradient is expelled from the bulk and concentrated in the boundary layers. An analysis along the lines of (7) shows that in the center of the cell one expects exponentials of width D rather than $D^{1/2}$ as in (5) because of the reduction in the local mean gradient.

Our derivation of (1) was frankly heuristic and phenomenological, and since we are unable to be more systematic it is perhaps helpful to mention a model of a passive scalar on a one-dimensional lattice with spacing ξ whose computer solution agrees well with (1).[13] The advection is modeled by an instantaneous interchange, "flip," of the current values of θ_i and θ_{i+1} with a rate per site $\sim 1/\tau$, that is uniform in space and time. The diffusion is included exactly by solving

$$\partial_t\theta_i=\kappa(\theta_{i+1}+\theta_{i-1}-2\theta_i)+\text{"flip"} \qquad (11)$$

with $\kappa\sim 1/\tau$ so as to model the eddy diffusivity. When a mean gradient is imposed on (11) the steady-state PDF calculated numerically is purely exponential. Note that paradoxically the diffusion (advection) alone will give a PDF that tends to Gaussian with a variance that decreases (increases) in time. If κ is incorrectly taken as the molecular diffusivity, $\kappa\ll 1/\tau$, then (11) gives a Gaussian core and exponential tails.

It may seem paradoxical to the reader that Gaussian distributions are so elusive within our class of models.[14] They arise from the sum of uncorrelated terms and would appear within a closure model for the large scales of turbulence since, in addition to the eddy damping, there is an additive, inhomogeneous, "force" F that maintains the scalar variance. It can be incorporated into (1) by composing $P_k(t+\tau)$ with a second transformation

$$P_k'\sim\left\langle\exp\left[ik\left(\theta+\int_0^\tau F\right)\right]\right\rangle \sim\exp\left[-k^2\left\langle\int F\int F\right\rangle\Big/2\right]P_k\,.$$

The Gaussian multiplying the $D\nabla^2$ terms does not qualitatively alter the solution and may be dropped. The only change to Eq. (4) is a factor $\exp(\sigma k^2)$, $\sigma=\langle\int F\int F\rangle$, so the tails are exponential since χ still has poles, and there is a well-defined Gaussian regime for smaller fluctuations if $\sigma\gg D(\partial_x\langle\theta\rangle)^2$.

An inhomogeneous term proportional to the random velocity times the mean gradient appears in (1) if we separate the field θ into its average and a fluctuation. This F, however, is not independent of the remaining nonlinear piece of the advection as was implicitly assumed in the previous paragraph when deriving the change in Eq. (4). A Gaussian PDF with exponential tails has recently been seen in several fluid simulations under conditions somewhat similar to those we have envisaged.[15] A definitive explanation has not yet appeared.

To understand how our theory might be applied to an experiment, it is instructive to consider the progression from the Lorentz model to Kolmogorov turbulence. When the dimension of the attractor is small, the probability distributions at all points in space are strongly correlated and reflect the invariant measure in *phase space* of the dynamical system. They will be neither exponential nor Gaussian. With more modes and the scalar injected via the walls in, for instance, a turbulent planar Couette flow, most of the scalar variance is associated with the large scales and $\theta(x)$ may be Gaussian since it is the sum of some reasonable number of roughly independent modes. Our model does not apply since P is correlated on the scale of the box.

Imagine now within a Kolmogorov picture that we high-pass filter the signal with cutoff $\omega=\Omega(\xi_\omega^{-1})$, where Ω is the characteristic straining frequency as a function of wave number for the turbulence. With ω intermediate between $\Omega(1)$ and the Kolmogorov cutoff frequency, one could imagine P decorrelated on a scale ξ_ω and apply (1). Simultaneously, a spacial average of θ over a region of size ξ could be done but is not essential since, within a cascade picture, the variance of θ at a point comes from the largest scale available which is ξ_ω after filtering.

When applied to "natural" turbulence as in Ref. 9, it is not fruitful to attempt to estimate the various parameters in (1) and (8) since the model is too idealized. Rather, the temperature variance in the bulk should be estimated directly and a fit made with (3) and (4). Within our highly schematic microscopic random advection model, there is no natural physical parameter that tunes F independently and that could create a well-defined Gaussian regime. From visualizations performed by directly manipulating the boundary layers,[16] Gaussians are associated with organized plumes which naturally translate into an enhanced F.

The most telling experimental exploration of our theory may come from "artificial" turbulence prepared either by randomly forcing a fluid back and forth through a grid with period ξ at moderate Reynold's number or in a long Taylor-Couette cell with a gap ξ and an axial scalar gradient. Provided there is no large-scale flow, the Peclet number is much bigger than 1, and the Prandtl number is not too large should one find a PDF with a velocity-independent variance $\sim\xi\, d\langle\theta\rangle/dx$, and that fits (3) and (4) where the boundary conditions define the gradient.

Our model illustrates a class of equations in which exponentials arise from a strip of analyticity in the Fourier-transformed PDF. Although the mathematics is rigorous, the physics remains obscure. The usual formula for Poisson statistics as a function of a continuous event number appears exponential for some range of parameters but ultimately is slightly steeper and with a Fourier transform that is entire because of the factorial. Gradient PDF's may fall off slower than exponential[17] and their transforms have no strip of analyticity.

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