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Construction of Multifractal Measures in Dynamical Systems from Their Invariance Properties

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We show that multifractal measures arising from the symbolic dynamics of chaotic systems can be reproduced using iterated-function systems involving Möbius maps. This powerful approximation scheme is exact for hyperbolic billiards: The coding measures of zero-angle N-sided polygonal billiards are *exactly* rendered by $N-1$ Möbius maps. We also approximate with success the coding measures for the anisotropic Kepler system.

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In two important works, Gutzwiller[1,2] has introduced and studied multifractal measures which arise in the coding of chaotic dynamical systems. These measures provide full information on the motion of the system under consideration, describe its phase-space structure, and play an important role in classical and semiclassical averages.[3,4] It is desirable to have a transparent construction of these dynamical measures, which, for instance, would allow a fast determination of periodic trajectories. In this Letter, we show that this goal can be achieved within the formalism of iterated-function systems (IFS),[5] in an exact (in the case of hyperbolic billiards[1]) or approximate form (for the anisotropic Kepler motion[2]).

To start with, let us consider the Series[6]-Gutzwiller example of geodesic motion in a singular triangle $\mathcal{T}$ in the hyperbolic upper half plane $[ds^2=(dx^2+dy^2)/y^2]$. $\mathcal{T}$ is bounded by the circle $\mathcal{R}$ of radius $\frac{1}{2}$, centered at $x=\frac{1}{2}$, and the lines $x=0$ (to be denoted by $\mathcal{O}$) and $x=1$ ($\mathcal{L}$). Geodesics in this space are given by the upper half of circles centered on the real axis. Their intersections with the real axis are denoted by ξ (the infinite past) and η (the infinite future) [see Fig. 1(a)]. Let $\xi<0$, $\eta>0$. A trajectory enters $\mathcal{T}$ from the side $x=0$, at time $t=0$, and exits from either the circle $\mathcal{R}$ or the vertical side $\mathcal{L}$. Periodic boundary conditions are imposed by mapping $\mathcal{R}$ onto $\mathcal{O}$ via $z\to z/(1-z)$ and $\mathcal{L}$ onto $\mathcal{O}$ via $z\to z-1$. The dynamical system so defined will be pictorially called a *hyperbolic billiard.* The symbolic code of a trajectory of such a billiard is the doubly infinite (past and future) sequence of symbols r,l, which records the intersections of the geodesic trajectory with $\mathcal{R}$ and $\mathcal{L}$, respectively. It will be indicated with $\{s_i\}$, $s_i=r,l$, $i=-\infty,\infty$. The symbolic code is in one-to-one

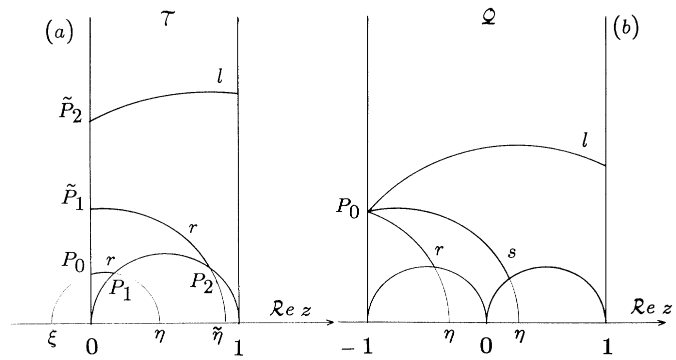

FIG. 1. (a) The hyperbolic triangular billiard $\mathcal{T}$. A geodesic in the hyperbolic upper half complex plane is shown as a semicircle orthogonal to the real axis. It originates from the point ξ at time $t=-\infty$ and ends at the point η at time $t=+\infty$. At time $t=0$ the motion leaves the point P_0 on the boundary of $\mathcal{T}$. It hits the boundary again at P_1: $s_0=r$ is added to the symbolic code, and the arc P_1-η is mapped to $\tilde{P}_1$-$\tilde{\eta}$ by $z\to z/(1-z)$. P_2 is the next intersection with the boundary $(s_1=r)$. (b) The hyperbolic quadrilateral billiard $\mathcal{Q}$. Three different trajectories leaving the point P_0 are coded by $s_0=r,s,l$ since they hit the sides of $\mathcal{Q}$ respectively on the right, straight in front, and on the left of P_0.

relation (modulo sets of zero measure) with a geodesic trajectory.[6]

With no loss of generality, let $\eta \le 1$, and let us consider only the future part of the code, $(s_0,s_1,\dots)$, where now $s_0=r$. It depends *only* upon the infinite future point η, in an arithmetical fashion. In fact, let $[n_1,n_2,\dots]$ be the continued-fraction (CF) expansion of η: $\eta=1/(n_1+1/(n_2+1/\cdots$. The desired relation is that the number of initial consecutive r's in the symbolic code equals n_1, the number of the following consecutive l's equals n_2, n_3 consecutive r's hence follow, and so on.[1] A *coding function F* can be associated with each trajectory [labeled by η, so that $F=F(\text{code})=F(\eta)$] in such a way that it is possible to uniquely recover the coding sequence from the value of the function. This is accomplished by defining $\varepsilon(r)=0$, $\varepsilon(l)=1$, and putting

$$F(\eta)=\sum_{i=1}^{\infty}\varepsilon(s_i)2^{-i}. \tag{1}$$

An equivalent formula for F in this case is $F=2^{-n_1+1}-2^{-n_1-n_2+1}+2^{-n_1-n_2-n_3+1}-\cdots$. Knowledge of the function $F(\eta)$ is equivalent to the full information on the symbolic dynamics of this chaotic system. The multifractal properties of F (shown in Fig. 2) have been numerically investigated in Ref. 1. We now show that they follow from a semigroup of exact symmetries which can be equivalently used to characterize and define this function.

In fact, it is convenient to study three mappings defined on the CF expansion of any number between 0 and 1,

$$\begin{aligned} &M_0: [n_1,n_2,\dots]\to[1,n_1-1,n_2,\dots],\\ &M_1: [n_1,n_2,\dots]\to[n_1+1,n_2,\dots],\\ &M_2: [n_1,n_2,\dots]\to[1,1,n_1-1,n_2,\dots].\end{aligned} \tag{2}$$

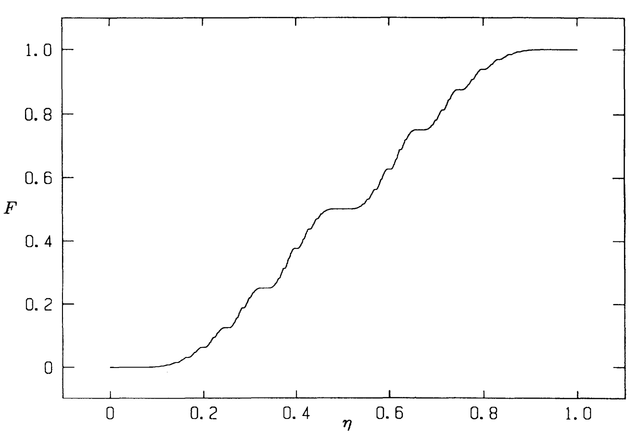

FIG. 2. Coding function for the motion in the arithmetic triangular billiard $\mathcal{T}$. The reader can verify its similarity properties: e.g., mapping η to $M_1(\eta)$ one obtains over $[0,\frac{1}{2}]$ the graph of F, scaled by a factor of $\frac{1}{2}$.

(Possible occurrences of $n_k=0$ are obviated by identifying $[\dots,n_{k-1},0,n_{k+1},\dots]=[\dots,n_{k-1}+n_{k+1},\dots]$.) Setting $x=1/(n_1+1/(n_2+1/\cdots$, Eqs. (2) entail three Möbius mappings of the unit interval into itself:

$$\begin{aligned} M_0(x)&=1-x,\\ M_1(x)&=x/(1+x),\\ M_2(x)&=1/(2-x).\end{aligned} \tag{3}$$

Three linear mappings, $P_0(y)=1-y$, $P_1(y)=y/2$, and $P_2(y)=(y+1)/2$, can be paired with the former transformations in such a way that one obtains the functional relations

$$F(M_i(\eta))=P_i(F(\eta)),\quad i=0,1,2, \tag{4}$$

which can be proved by direct substitution in Eqs. (1) and (2). These equations represent mathematically the scaling properties of the curve $F(\eta)$. They can be extended to the full semigroups generated by M_i and P_i. We also note that the action of the M_i's on $x=1$ generates the Farey tree. We now show that the smaller semigroups generated only by M_1,M_2 and P_1,P_2 are sufficient to completely define F. In fact, Eqs. (4) are of the "fractal-generating" form investigated by Dubuc,[7] and can be cast into the formalism of iterated-function systems. To do this, let us consider the mappings w_i from the unit square into itself given by $w_i(x,y)=(M_i(x),P_i(y))$, for $i=1,2$. Since these mappings are contractive (in the Euclidean metric), they possess a *unique* invariant set $G=\cup_i w_i(G)$. The *graph* of the function F, i.e., the sets of points $(x,F(x))$, satisfies such invariance as a result of Eqs. (4), and hence coincides with G.

We recall that G is also the *attractor* of the dynamical system obtained by successive (random) iteration of the maps w_i:[5] Let us take an arbitrary point (x_0,y_0) in the unit square [e.g., (0,0), which we know to belong to G], and map it via the IFS w_i, $i=1,2$:

$$(x_1,y_1)=w_\sigma(x_0,y_0)=(M_\sigma(x_0),P_\sigma(y_0)), \tag{5}$$

where σ is a random variable which takes the values 1 and 2, with equal probability. Repeating this operation over and over, $(x_n,y_n)=w_\sigma(x_{n-1},y_{n-1})$, we get a set of points (x_n,y_n) which lie upon and fill up the set G. This allows a fast computation of F.

Single trajectories (in particular, periodic trajectories) can also be easily retrieved once F is known: Let q be the coding number computed via Eq. (1) from the symbolic definition of a particular trajectory. The value of η corresponding to such an unknown trajectory is then easily retrieved via the equation $F(\eta)=q$, which can be solved to arbitrary precision.

The function $F(\eta)$ is best seen as the distribution function of measure. A binomial multifractal measure[1,8] can model F; besides this, a new IFS characterization can be given as follows. In contrast with the previous two-dimensional IFS, we introduce a one-dimensional IFS generated by the maps M_i, $i=1,2$, restricted to the unit interval. Its balanced measure $d\mu$ is defined by the property $d\mu=\frac{1}{2}[d\mu(M_1^{-1})+d\mu(M_2^{-1})]$. As as result of Eqs. (4), one identifies $d\mu=dF$. As a consequence, the one-dimensional random process analogous to (5), $x\to M_\sigma(x)$ (again, σ is a random variable taking the values 1 and 2, with equal probability), has dF as its invariant measure. This property can be used to compute integrals in dF.

Let us now consider the Hölder exponent $\alpha(\eta)$ of $F(\eta)$: $F(\eta+h)-F(\eta)\sim h^{\alpha(\eta)}$, as $h>0$ tends to zero. Because of Eqs. (4), and the fact that M_i are smooth maps, one finds the crucial relation $\alpha(\eta)=\alpha(M_i(\eta))$: $\alpha(\eta)$ is therefore invariant under the semigroup of Möbius transformations generated by M_i, $i=0,1,2$. In particular, whenever α takes a value $\bar{\alpha}$, say, at $\bar{\eta}$, then the same value will be taken at all the forward iterates of $\bar{\eta}$, which are usually dense in [0,1]: As $\alpha(0)=\infty$, we have $\alpha(\eta)=\infty$ at all rational values.

The semigroup relation above can be extended to the full group (which turns out to be equivalent to the modular group) for a modular function $\tilde{\alpha}$, defined on the full real line, which coincides with α on [0,1]. The interesting multifractal properties found in Ref. 1 appear therefore to be of number-theoretical origin.

The formalism introduced here is also suited for more general cases. Besides the singular triangle $\mathcal{T}$, we now consider the billiard identified by a quadrilateral $\mathcal{Q}$ (Ref. 1) whose sides are (1) the vertical $x=-1$ line, (2) the semicircle of radius $\frac{1}{2}$ around $x=-\frac{1}{2}$, (3) the semicircle of radius $\frac{1}{2}$ around $x=\frac{1}{2}$, and (4) the vertical $x=1$ line [see Fig. 1(b)]. Side (1) is identified with (3) via the transformation $A=[\delta,\delta;\delta,(1+\delta^2)/\delta]$ (the short notation $[a,b;c,d]$ defines the Möbius map $x\to(ax+b)/(cx+d)$). Also, sides (4) and (2) are linked by $B=[\delta,-\delta;-\delta,(1+\delta^2)/\delta]$. Here, δ is a real parameter. The coding of the geodesic motion on this torus is obtained by the letters r,s,l, which are assigned at each intersection with the boundary: r indicates that one hits the side while turning on the *right*, s means *straight*, and l means *left*. The future part of the coding depends only upon the infinite future point $\eta\in[0,1]$.

A generalization of Eq. (1) to a three-letter alphabet, with $\varepsilon(r)=0$, $\varepsilon(s)=1$, and $\varepsilon(l)=2$, enables us to define a new function F, uniquely associated with η and the dynamics. As in the previous case, an IFS reconstruction of this function, and of its invariant measure, is possible (Fig. 3). The required Möbius transformations are here in the number of three, $M_1=[-\delta,\delta;-\delta,(1+\delta^2)/\delta]$, $M_2=[\delta,\delta;\delta,(1+\delta^2)/\delta]$, and $M_3=[1-2\delta^2,2\delta^2;-2\delta^2,1+2\delta^2]$. They pair with the linear maps $P_1(y)=(1-y)/3$, $P_2(y)=(1+y)/3$, and $P_3(y)=(2+y)/3$ to produce the fundamental invariant relations (4), which now hold for $i=1,2,3$. The IFS properties shown previously extend immediately to this case, and, more generally, to N-sided hyperbolic billiards with corners at infinity,

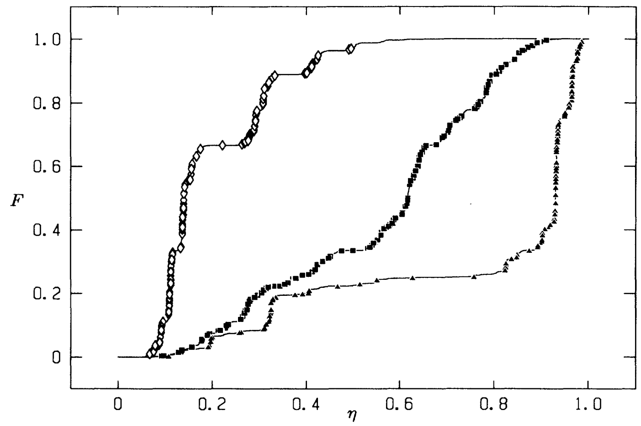

FIG. 3. Coding functions for the motion in the hyperbolic quadrilateral billiard $\mathcal{Q}$, for different values of δ (continuous lines). Diamonds ($\delta=1/\sqrt{7}$), squares ($\delta=1$), and triangles ($\delta=\sqrt{7}$) are drawn as points on the attractor of the corresponding IFS.

whose coding measures can be rendered exactly by $(N-1)$-Möbius-map iterated-function systems.

Let us now turn to another important example, the anisotropic Kepler problem,[2] defined by the Hamiltonian

$$H=\frac{p_x^2}{2\mu}+\frac{p_y^2}{2\nu}-\frac{1}{(x^2+y^2)^{1/2}}, \tag{6}$$

where the effective masses μ and ν are different. This system is effectively chaotic when the mass ratio μ/ν is sufficiently high (≥ 5). A symbolic coding for this system has been also proposed by Gutzwiller as follows.[9] With no loss of generality the value of the constant-energy surface can be fixed, e.g., $H=-\frac{1}{2}$. Let us then take a set of trajectories starting on the x positive axis, with zero initial p_x momentum. From Eq. (6) one sees that this specifies a unique trajectory, for any $0\leq x\leq 2$. Following the time evolution of such trajectories, one records a bit sequence b_j determining the signs of the future intersections of the trajectory with the x axis; $b_i=0$ if the ith intersection occurs for $x\leq 0$, and $b_i=1$ otherwise. A coding function F is then defined via $F(x)=\sum b_i 2^{-i}$. As for billiards, this function is nondecreasing, and shows multifractal features. We thereby attempt to approximate it via Möbius-map IFS coding transformations: Suitable coefficients of new Möbius maps M_i and linear maps P_i are found so that Eqs. (4) (and the associated IFS) define a good approximation of F.

This approximation can be done, for instance, as follows.[10] One computes directly (from the equations of motion) a finite (small) number of values of F, $F(x_n)=y_n$. One then requires that the attractor G of the IFS to be determined exactly interpolates the points (x_n,y_n). This leads to a set of equations for the coefficients of

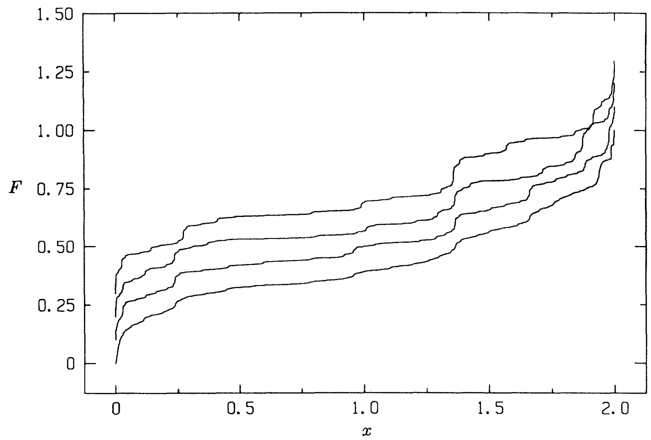

FIG. 4. Exact coding functions for the anisotropic Kepler problem, for $\mu/\nu=3$ (lowest curve). The approximated curves obtained by M-map IFS have been shifted in the F direction for clarity. From top to bottom: $M=2$ (shift=0.3), $M=4$ (shift=0.2), $M=8$ (shift=0.1).

M_i, P_i which are readily solved. This is a particular instance of the general problem of approximation of fractal measures discussed in Refs. 11 and 12.

In Fig. 4 we show the coding curve with mass ratio $\mu/\nu=3$, and a few Möbius-map IFS approximations. Using more maps steadily increases the quality of the approximation. No particular characteristics of the anisotropic Kepler motion have been adopted in our reconstruction, which can be easily shown to converge for any function F.[12] Hence, this method can be successfully applied to any chaotic system, provided its symbolic dynamics is known explicitly.

The multifractal measures studied in this paper are transformations from the *trivial* coding given by the initial coordinates to the *dynamical* coding induced by appropriate partitions of the phase space. They therefore contain the complete information on the dynamics. The formalism of IFS applied here to describe these measures renders explicit their fractal structure, and provides us with analytically treatable, significant models which owing to their simplicity can be studied *in place* of the original system. We have shown that this approximation is exact and quite informative for hyperbolic billiards, and that it can be adapted arbitrarily well to any chaotic system, whose symbolic dynamics is known explicitly. We are investigating the efficacy of the same formalism when exact generating partitions are not explicitly known.

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