Spectral Bandwidth in Free-Electron-Laser Oscillators

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(Received 23 January 1991)

The evolution of the spectral bandwidth in free-electron-laser (FEL) oscillators is studied and found to be different for different saturation characteristics: For weakly saturated FELs in storage rings, the limiting bandwidth is given by that derived in the supermode theory; for linac-based FELs, where the gain reduction due to the high-intensity effect is significant, it is determined by the Fourier transform of the electron pulse length; for the case of a dc beam, it is given by a Schawlow-Townes formula, but the approach to the limit is very slow.

PACS numbers: 42.55.Tb, 42.10.Mg, 42.60.Gd

There has been some confusion in the past about the achievable bandwidth in free-electron-laser (FEL) oscillators.[1] For FELs driven by electron-beam micropulses, the bandwidth is usually assumed to be given by the so-called Fourier-transform limit, i.e., the ratio of the radiation wavelength and the electron-beam bunch length. However, the analysis in terms of the supermode theory leads to a different formula, given by a geometric average of the gain bandwidth and the transform-limited bandwidth.[2-4] Extension of the Schawlow-Townes limit[5] to FELs has also been discussed.[6,7] Experimentally, the bandwidth of the storage-ring-based FELs in the visible region is consistent with the prediction of the supermode theory. For the linac-based infrared FELs constructed so far, the bandwidth appears to be given by the transform limit, although the situation is somewhat ambiguous because the bandwidths calculated from two formulas are not very different. It is thus important to resolve the bandwidth issue in FELs. In this Letter, we study the evolution of the spectral and temporal profiles in FELs in terms of a simple but physically reasonable model, and find that the limiting bandwidth is given either by the Fourier transform or by the supermode theory depending on the particular saturation characteristics.

We show that the bandwidth of the optical pulse narrows as $1/\sqrt{n}$ as the number of passes n of the electron beam through the optical cavity increases. The temporal width also narrows in a similar fashion in the beginning of the intensity buildup. This, together with the fact that the product of the temporal and the spectral widths must be greater than a minimum value, leads to the limiting bandwidth predicted by the supermode theory. The supermode theory is valid for a weakly saturated system such as storage-ring-based FELs, where the gain can be regarded as a constant.

For linac-based FELs, however, the optical power evolves to a level where the reduction of gain due to high intensity, i.e., the gain saturation, becomes important. Observing that the gain saturation is homogeneous in frequency but inhomogeneous in time, we derive that the limiting bandwidth is then given by the Fourier transform of the electron pulse length. In this discussion, we find it necessary to distinguish between the intensity saturation and the "spectrum saturation"; the time to reach the limiting spectrum is typically longer than the time to reach the intensity saturation.

For the case of a dc electron beam, the narrowing of the bandwidth continues until it reaches a small value determined by the spontaneous radiation, the Schawlow-Townes limit. However, as the bandwidth narrowing is slow, $1/\sqrt{n}$, it takes a long time to reach this value, typically a day or longer.

To begin the analysis, let $dP(\omega,\tau;n)/d\omega$ be the τ-dependent spectral density of the optical power at the beginning of the nth passage, and $dS(\omega,\tau)/d\omega$ a similar quantity for the spontaneous radation emitted in one pass. Here, ω is the frequency and $c\tau$ (c the speed of light) is the distance from the pulse center. Since ω and τ are conjugate variables under Fourier transformation, the following inequality must be satisfied:

$$\sigma_\omega \sigma_\tau \geq \tfrac{1}{2}\,, \tag{1}$$

where σ_ω and σ_τ are the rms values of the spectral and the temporal widths. The quantities $dP/d\omega$ and $dS/d\omega$ should, strictly speaking, be understood as the Wigner distribution.[8] However, when $\sigma_\omega \sigma_\tau \gg 1$, the distributions $dP/d\omega$ and $dS/d\omega$ can be interpreted as local spectral densities at τ. With this interpretation, a simple model for the evolution of the optical power may be written as follows:

$$\frac{d}{dn}\left(\frac{dP(\omega,\tau;n)}{d\omega}\right) = [g(\omega,\tau;n)-\alpha]\frac{dP(\omega,\tau;n)}{d\omega} + \frac{dS}{d\omega}\,. \tag{2}$$

Here g is the gain parameter and α is the total loss per round trip. Equation (2) is the statement that the increase of the optical power during the nth passage of the electron beam consists of two terms, that due to the amplification of the power already present and that due to the spontaneous radiation. In the following, we solve Eq. (2) to find the evolution of the quantity $dP/d\omega$, regarding ω and τ to be independent. The rms widths σ_ω

and σ_τ are calculated using $dP/d\omega$ as the weight function. As the signal develops coherence from the initial spontaneous radiation, the product $\sigma_\omega\sigma_\tau$ becomes smaller than a large initial value corresponding to the incoherent spontaneous radiation. The limit is reached when Eq. (1) becomes an equality for a certain n, beyond which Eq. (2) must be regarded as inapplicable. For an explicit solution, we need to consider the cases of storage rings and linacs separately as the behavior of the gain function $g(\omega,\tau;n)$ is different.

First we consider the case of the storage-ring-based FELs, in which the saturation is due to the induced energy spread and bunch lengthening of the electron beams that accumulate from pass to pass. The power level at saturation is determined by a balance between the inhomogeneous gain reduction and radiation damping, and is given by the Renieri limit.[9] It is well below the level at which particle trapping in the ponderomotive potential becomes significant. Therefore, we may assume that the gain is independent of the optical power and n, as follows:

$$g(\omega,\tau;n)=g_0F(\omega)T(\tau)\,. \tag{3}$$

The function $F(\omega)$ describes the frequency dependence of the gain. For frequencies near the resonance frequency ω_0,

$$F(\omega)=1-\frac{(\omega-\omega_0)^2}{2\sigma_N^2\omega_0^2},\quad \frac{|\omega-\omega_0|}{\omega_0}<\sigma_N\,. \tag{4}$$

In the above, σ_N is the gain bandwidth, given approximately by $\sigma_N\sim 1/2N$, where N is the number of undulator periods. The function $T(\tau)$ describes the temporal profile of the electron pulse. For τ near the pulse center, it is of the form

$$T(\tau)=1-(\tau/2\sigma_{\tau 0})^2\,. \tag{5}$$

Here $\sigma_{\tau 0}$ is the rms bunch length of the electron pulse in time.

With the gain function specified by Eq. (3), Eq. (2) can be solved easily. The result is

$$\frac{dP(\omega,\tau;n)}{d\omega}=\frac{\exp\{[g_0F(\omega)T(\tau)-\alpha]n\}-1}{g_0F(\omega)T(\tau)-\alpha}\frac{dS}{d\omega}\,. \tag{6}$$

In view of Eqs. (4) and (5), Eq. (6) implies that the spectral width and the temporal width of the optical pulse become narrower as the number of passes n increases, as follows:

$$\frac{\sigma_\omega}{\omega_0}=\sigma_N\left(\frac{1}{g_0n}\right)^{1/2},\quad \sigma_\tau=\sigma_{\tau 0}\left(\frac{1}{g_0n}\right)^{1/2}. \tag{7}$$

The simultaneous narrowing in the spectral width and the temporal width, Eq. (7), must stop, to be consistent with the inequality (1). This occurs for $n\geq n_c=2\pi c\sigma_{\tau 0}/g_0\lambda N$. The limiting bandwidth for this n_c is

$$\frac{\sigma_\omega}{\omega}=\left(\frac{1}{2N}\frac{\lambda}{4\pi c\sigma_{\tau 0}}\right)^{1/2}. \tag{8}$$

This is a geometric average of the gain bandwidth and the transform-limited bandwidth $\lambda/4\pi c\sigma_{\tau 0}$, and was first derived in the context of the supermode theory.[2,3] It appears to be consistent with the results of FEL experiments in storage rings[3,4] (with a suitable replacement of $1/2N$ by a factor appropriate for optical klystrons).

Next we consider the case of the linac-driven FELs, in which the optical pulse interacts with a fresh electron bunch in each round trip. The particle trapping in the ponderomotive potential becomes significant, and the FEL intensity reaches saturation when the electron motion in the undulator corresponds to about one-half of the synchrotron oscillation period. In this case, it is necessary to take into account the gain reduction caused by the high-intensity effect. A simple way to model the gain reduction is to replace Eq. (3) by

$$g(\omega,\tau;n)=\frac{g_0F(\omega)T(\tau)}{1+P(\tau;n)/\bar{P}}\,, \tag{9}$$

$$P(\tau;n)=\int\frac{dP(\omega,\tau;n)}{d\omega}d\omega\,. \tag{10}$$

In Eq. (9), $\bar{P}$ is a parameter which sets the scale of the saturation intensity; it is about the power at which electrons undergo a one-half-period synchrotron oscillation in passing through the undulator. The precise relation between $\bar{P}$ and the saturation intensity P_s is derived later in Eq. (15); $P_s=[(g_0-\alpha)/\alpha]\bar{P}$. In certain cases where the oscillator saturates at high powers, the spectrum could exhibit sidebands and chaotic behavior. In the following, we assume that the sideband development is suppressed by suitable means, such as the cavity detuning.[10]

According to Eq. (9), the gain reduction for a given frequency ω and temporal position τ is determined by a sum of the optical intensities over all ω, but evaluated at the same τ. Thus, Eq. (9) is a model for gain saturation which is homogeneous in ω but inhomogeneous in τ.[11] The gain saturation is homogeneous in ω because all frequency components which lie within the gain bandwidth should contribute equally to the saturation. [Strictly speaking, the integral in Eq. (10) should be replaced by an integral which extends to a frequency region within the gain bandwidth. However, we are mainly interested in the cases for which the spectral width of the function $dP/d\omega$ is much narrower than the gain bandwidth, in which case Eq. (10) is a good approximation.] On the other hand, the saturation is inhomogeneous in τ since optical intensities at two τ's separated by more than one slippage distance $N\lambda/c$ should evolve independently.[12] Here we are assuming, as is almost always the case, that the pulse is much longer than the slippage distance. We are also assuming that $\delta L\ll N\lambda$, where δL is the amount by which the cavity length is shorter than that required for an exact synchronism between the optical pulses and the electron pulses. In general, it is advantageous to choose the so-called cavity detuning δL different from zero to suppress the sideband development.[10] However, the detuning required for that purpose is typically about $\delta L \lesssim 0.1N\lambda$ when the gain is less than 100%.[13] Thus, the above inequality is usually well satisfied.

Equations (9) and (10) are intended to represent a heuristic but qualitatively correct model for the saturation effect which is homogeneous in ω and inhomogeneous in τ. The quantitative details of the saturation effect could be different from this model.[14] However, the basic conclusions of this Letter remain unaffected by such details.

Before solving Eqs. (2) and (9) explicitly, we discuss the main features of the spectrum evolution in linac-based FELs qualitatively as follows.

In the beginning of the FEL evolution the ratio $P(\tau;n)/\bar{P}$ is small so that Eq. (9) reduces to Eq. (3). Therefore, the spectral width and the temporal width will both start to narrow as in Eq. (7). As the optical power increases, the gain becomes smaller due to the intensity-dependent effect. This so-called saturation effect takes place first at $\tau=0$ where the initial gain is highest. However, the optical intensities at $\tau\neq 0$ will keep increasing until they reach their own saturation level. Thus, the temporal width of the optical pulse, after initial narrowing, will broaden as the optical intensity approaches the saturation level, and eventually becomes the same as the width of the electron beam. The limiting bandwidth in this case is therefore determined by the Fourier transform of the electron pulse profile:

$$\sigma_\omega/\omega = \lambda/4\pi c\sigma_{\tau 0}\,. \tag{11}$$

Equation (11) will be referred to as the Fourier-transform-of-the-electron-pulse-limited, or simply the transform-limited, bandwidth and is consistent with the results of the FEL experiments in linacs.[15]

For an explicit solution we proceed as follows: We integrate Eq. (2) with respect to ω. In doing so, we assume that the width of the optical spectrum is much narrower than the gain bandwidth so that

$$\int d\omega\, F(\omega)\frac{dP(\omega,\tau;n)}{d\omega} \approx \int \frac{dP(\omega,\tau;n)}{d\omega}d\omega = P(\tau;n)\,. \tag{12}$$

We obtain

$$\frac{d}{dn}P(\tau;n) = [g(\tau;n)-\alpha]P(\tau;n)+\Delta S\,, \tag{13}$$

where $\Delta S = \int d\omega(dS/d\omega)$ is the total spontaneous power, $g_0(\tau;n) = g_0(\tau)/[1+P(\tau;n)/\bar{P}]$, and $g_0(\tau)=g_0T(\tau)$. Let $\varepsilon = \Delta S/P$, which is a very small number, typically 10^{-8} or less. If $g_0T(\tau) < \alpha-\varepsilon$, the FEL is below threshold, and $P(\tau;n)$ is of the order of ΔS for all n. On the other hand, if $g_0T(\tau) > \alpha-\varepsilon$, the solution of Eq. (13) is

$$\frac{[P_0(\tau)+P(\tau;n)]^{\kappa-1}}{[P_s(\tau)-P(\tau;n)]^{\kappa+1}} = \frac{[P_0(\tau)]^{\kappa-1}}{[P_s(\tau)]^{\kappa+1}}e^{2\alpha n}\,, \tag{14}$$

where

$$\kappa = \frac{g_0(\tau)+\alpha}{g_0(\tau)-\alpha}\,,\quad P_0(\tau) = \frac{\Delta S}{g_0(\tau)-\alpha}\,,$$
$$P_s(\tau) = \frac{g_0(\tau)-\alpha}{\alpha}\bar{P}\,. \tag{15}$$

The limiting cases of Eq. (14) are

$$P(\tau;n) = \begin{cases} P_0(\tau)(e^{[g_0(\tau)-\alpha]n}-1), & n \ll n_s, & (16)\\ P_s(\tau)\{1-[P_s(\tau)/P_0(\tau)]^{\alpha/g_0(\tau)}e^{-\alpha[1-\alpha/g_0(\tau)n]}\}, & n \gg n_s\,. & (17)\end{cases}$$

In the above, $n_s = [1/g_0(\tau)]\ln[P_s(\tau)/P_0(\tau)]$ is the number of passes characterizing the saturation of the power. The optical power $P(\tau;n)$ is practically constant at $P_s(\tau)$ for $n \geq n_s$. From Eq. (16), we see that the temporal width begins to narrow at small n. However, the temporal profile at saturation is, assuming $g_0 \gg \alpha$, described by the electron density profile $T(\tau)$, the corresponding bandwidth being given by Eq. (11).

The behavior of $dP(\omega,\tau;n)/d\omega$ at large n is approximately given by the solution of the homogeneous part of Eq. (2) as follows:

$$\frac{dP(\omega,\tau;n)}{d\omega} = \frac{dP(\omega,\tau;0)}{d\omega}e^{G(\tau;n)F(\omega)-\alpha n}\,, \tag{18}$$

where $G(\tau;n) = \int_0^n g(\tau;n)dn$. It can be shown that

$$\sigma_\omega/\omega_0 = \begin{cases} \sigma_N(1/g_0 n)^{1/2}, & n \ll n_s\,, & (19)\\ \sigma_N(1/\alpha n)^{1/2}, & n \gg n_s\,. & (20)\end{cases}$$

The bandwidth will keep narrowing as described by Eqs. (19) and (20) until it reaches the Fourier-transform (of the electron beam) limit given by Eq. (11). Typically, the bandwidth after n_s passes $[\sim\sigma_N/(g_0 n_s)^{1/2}]$ is still broader than the Fourier-transform limit. Therefore the spectrum of optical pulses in an FEL cavity keeps evolving after the intensity reaches saturation at around $n=n_s$. By comparing Eqs. (11) and (20), we determine the number of passes n_ω required to reach the spectrum saturation as follows:

$$n_\omega = (1/\alpha)(4\pi c\sigma_{\tau 0}/2N\lambda)^2\,. \tag{21}$$

Finally, we consider the case of a dc electron beam. We may delete all τ dependence and replace $g(\omega,\tau;n)$ by $g(n)F(\omega)$ in Eq. (2). The steady-state solution as n approaches infinity is obtained by setting the right-hand side of Eq. (2) to zero. The limiting value of $g(n)$ for large n, $\bar{g}$, must be very nearly equal to α. Thus, we write $\bar{g}=\alpha(1-\delta^2/2)$, where $\delta \ll 1$. The sign of the δ^2 term here is chosen so that the limiting spectral density,

$$\frac{dP_s}{d\omega}=\frac{1}{\alpha[\delta^2/2+(\omega-\omega_0)^2/2\sigma_N^2\omega_0^2]}\left(\frac{dS}{d\omega}\right), \quad (22)$$

is a positive-definite quantity. By integrating Eq. (22) (neglecting the ω dependence of $dS/d\omega$), we determine

$$\delta=2\pi\frac{\Delta S}{P_{\text{gen}}}, \quad \Delta S=\sigma_N\omega_0\frac{dS}{d\omega}\approx\int d\omega\frac{dS}{d\omega}, \quad (23)$$

where $P_{\text{gen}}=\alpha P_s$ is the generated power. Thus, the limiting distribution is a Lorentzian with

$$\Delta\omega/\omega|_{\text{FWHM}}=2\pi\sigma_N\Delta S/P_{\text{gen}}. \quad (24)$$

Equations (23) and (24) are similar to the Schawlow-Townes formulas[5-7] except for the replacement of the bandwidth of the optical cavity by the gain bandwidth σ_N. Typically, the limiting bandwidth is smaller at least by a factor of 10^6 compared to σ_N. To reach this bandwidth via the gain narrowing described by Eq. (20) will take at least 10^{12} passes, which corresponds to about 1 day with a 10-m optical cavity. Single-mode operation of an FEL with a bandwidth similar in magnitude to that given by Eq. (24) has been reported.[16] However, the result is controversial experimentally[17] and unlikely theoretically because of the slow approach to the limiting bandwidth. A nonlinear mode-competition theory of FELs has been developed based on a third-order perturbation expansion to show that a single mode will eventually dominate by suppressing all other modes.[18] However, the theory is based on the interaction of a small number of modes, and does not treat the evolution from the initial multimode state. The approach to frequency saturation in long-pulse FELs was also studied numerically in Ref. 19, where the statistical effect of spontaneous radiation is included.

I thank A. N. Vinokurov for helpful discussions. An early work on the spectral evolution of FELs can be found in Ref. 20. This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

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