Evolution of Vortex Statistics in Two-Dimensional Turbulence

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Freely evolving two-dimensional turbulence is dominated by coherent vortices. The density of these vortices decays in time as $\rho \sim t^{-\xi}$ with $\xi \approx 0.75$. A new scaling theory is proposed which expresses all statistical properties in terms of ξ. Thus the average circulation of the vortices increases as $t^{\xi/2}$ and their average radius as $t^{\xi/4}$. The total energy is constant, the enstrophy decreases as $t^{-\xi/2}$, and the vorticity kurtosis increases as $t^{\xi/2}$. These results are supported both by numerical simulations of the fluid equations and by solutions of a modified point-vortex model.

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Because of its geophysical and astrophysical importance, the emergence and evolution of coherent vortices in freely evolving two-dimensional turbulence has been a subject of intense study in the last ten years.[1-6] In this Letter, we formulate a new scaling theory and present evidence for scaling behavior in both two-dimensional turbulence and a simple, punctuated-Hamiltonian, dynamical model of coherent vortices. In anticipation of this work on vortex dynamics, we have previously performed studies of punctuated-Hamiltonian models of aggregation in one-dimensional systems.[7]

The fluid-dynamical equations are

$$\zeta_t + J(\psi,\zeta) = \nu_p(-1)^{p+1}\nabla^{2p}\psi, \quad \zeta \equiv \nabla^2\psi, \tag{1}$$

where ψ is the stream function, ζ is the vorticity, $J(a,b) \equiv a_x b_y - b_x a_y$ is the Jacobian, $\nabla^2 \equiv \partial_x^2 + \partial_y^2$ is the Laplacian, and ν_p is the hyperviscosity for p a positive integer ($p=2$ here). The domain is a square of side $2\pi L$ and the boundary conditions are periodic in both x and y. Numerical solutions of Eq. (1) show that well separated, almost axisymmetric, coherent vortices emerge from structureless initial conditions. Between the vortices there is a background sea of small-scale, incoherent vorticity. After the emergence of the vortices the dynamics appears to be dominated by two processes: (1) mutual advection of well separated vortices in which Hamiltonian point-vortex dynamics is a good approximation,[2] and (2) merger of like-sign vortices during close encounters.[8] As a result of the mergers, the vorticity is concentrated in increasingly larger, fewer, and more widely separated vortices as time increases. A "vortex census" shows that, for a broad class of initial conditions, the number of vortices per area $\rho(t)$ decreases according to $\rho \sim t^{-\xi}$,[9] with ξ approximately 0.75.[6] The focus of this Letter is on the scaling properties of this "dilute vortex gas."

There is a simple dimensional argument that predicts $\xi = 2$. The kinetic energy per area $\mathcal{E}$ is invariant as $\nu_p \rightarrow 0$,[10] where

$$\mathcal{E} \equiv \frac{1}{4\pi^2 L^2}\int \tfrac{1}{2}\nabla\psi\cdot\nabla\psi\, d\mathbf{x} = \int d\mathbf{x}\int d\mathbf{x}'\,\zeta(\mathbf{x})G(\mathbf{x},\mathbf{x}')\zeta(\mathbf{x}'), \tag{2}$$

and G is the Green function for ∇^2. If $\mathcal{E}$ provides the only length scale in the problem, then, on dimensional grounds, one must have $\rho \sim 1/\mathcal{E}t^2$. Other authors[6,11] have noted the analogy with the kinetic theory of colloidal aggregation, which leads to the conclusion that $\rho \sim t^{-1}$. Both of these results clearly disagree with the turbulence simulations.

If we characterize the flow as a dilute vortex gas with density ρ and typical radius a and vorticity extremum ζ_{ext}, then we can express the conserved energy by the scaling relation

$$\mathcal{E} \sim \rho\zeta_{\text{ext}}^2 a^4. \tag{3}$$

This is easily seen as an approximation to the second expression in (2) for both spatial arguments within the same vortex (the self-energy $\mathcal{E}_s$).[12] It also characterizes the contribution to $\mathcal{E}$ from arguments in separate vortices (the configuration energy $\mathcal{E}_c$); $\mathcal{E}_c$ scales with the number of vortices, rather than the number of pairs, due to cancellations from vortices of opposite sign. We assume the contributions in (2) from arguments outside the vortices are negligible.

Inviscid dynamics [$\nu_p = 0$ in Eq. (1)] conserves vorticity on every fluid parcel; this suggests that some quantity related to the initial vorticity field affects the long-term dynamics, and this can be the only explanation for the failure of the $\rho \sim 1/\mathcal{E}t^2$ scaling. Turbulence solutions show that the average extremum is approximately conserved.[6] We therefore choose ζ_{ext} as the second invariant of our scaling theory. It is initial extrema that form the cores of the emerging vortices, and the cores are the locations of least dissipation during the subsequent evolution. Even during merger events, in which both participating vortices shed circulation into the background cascade, the larger of the vorticity extrema is preserved as the extremum of the new core.[13]

Given $\mathcal{E}$ and $\zeta_{\rm ext}$ there is both a length, $l \equiv \sqrt{\mathcal{E}}/\zeta_{\rm ext}$, and a time scale, $\tau \equiv 1/\zeta_{\rm ext}$. Dimensional reasoning only tells us that $\rho = l^{-2} g(t/\tau)$. Now we suppose that $g \sim t^{-\xi}$ and relate all scaling exponents to ξ.

From Eq. (3), $a(t) \sim l(t/\tau)^{\xi/4}$. If there is no tendency toward clumping, the average separation between vortices r scales as $\rho^{-1/2}$ or $r(t) \sim l(t/\tau)^{\xi/2}$. Thus the fraction of the plane covered by vortices decreases as $(t/\tau)^{-\xi/2}$. The typical circulation of a vortex is $\Gamma \sim \zeta_{\rm ext} \times a^2 \sim \tau \mathcal{E} (t/\tau)^{\xi/2}$. The velocity of a vortex center u is determined by advection due to its neighbors. Thus, as one would anticipate, $u \sim \Gamma/r \sim \sqrt{\mathcal{E}}$. The amplitude of the stream function is $\psi \sim ru \sim \Gamma \sim \tau \mathcal{E} (t/\tau)^{\xi/2}$. Using similar arguments, one finds the following scaling for the vorticity moments:

$$Z_n \equiv \frac{1}{4\pi^2 L^2} \int \zeta^n d\mathbf{x} \sim \rho \zeta_{\rm ext}^n a^2 . \qquad (4)$$

Thus the quadratic enstrophy decreases, $Z_2 \sim \tau^{-2} \times (t/\tau)^{-\xi/2}$, and the kurtosis of the vorticity distribution increases, $K_u \equiv Z_4/Z_2^2 \sim (t/\tau)^{\xi/2}$.

The preceding scaling relations describe a scenario which is consistent for any positive value of ξ. We test our scaling theory in two independent ways: first, by a comparison of the exponents obtained from a vortex census of a solution of Eq. (1) and then by a comparison of the same exponents obtained from a modified point-vortex model.

The statistics obtained from the solution of Eq. (1) are computed using the procedure described in Ref. 6. Figure 1 shows that our scaling theory correctly anticipates the connections between the various exponents. Our choice for the population decay exponent is $\xi = 0.75$. After an initial period of adjustment, $t^{\xi/2}$ and $t^{\xi/4}$ are good fits to the average vortex circulation magnitude Γ and average vortex radius a, respectively. The core vorticity extremum $\zeta_{\rm ext}$ does show a slight decrease towards the end of the calculation and we attribute this to the cumulative effects of the explicit diffusion on the right-hand side of Eq. (1). But apart from this slight frictional decay, t^0 is a convincing fit to the $\zeta_{\rm ext}$ data.

The integral moments predicted by the scaling theory also match the behavior in the turbulence solution. The velocity variance is very nearly constant with time, as expected from $\mathcal{E}$ invariance. Stream-function variance and vorticity variance and kurtosis match the scaling-theory predictions on long time scales, but they also exhibit sizable fluctuations about these trends.

Our second test of the scaling theory uses a modified point-vortex model (an example of a punctuated-Hamiltonian dynamical system). The particular form presented here is the simplest member of a hierarchy of models for structured turbulence which we are exploring. During most of the evolution, the model is the traditional, deterministic, chaotic Hamiltonian dynamics of point vortices,[14] where we assign each vortex a value for a and $\zeta_{\rm ext}$ that combine in the dynamically relevant Γ $(=\pi \zeta_{\rm ext} a^2)$. However, the evolution is punctuated by nonconservative transformations of the vortex population when two like-sign vortices come within a threshold separation distance of 3.3 times the average of the two radii.[15]

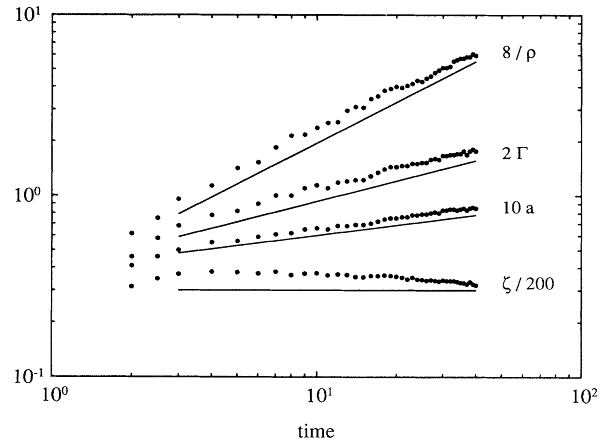

FIG. 1. Vortex data from a turbulence solution (Ref. 6): the reciprocal of the density of vortices ρ^{-1}, the mean absolute value of vortex circulation Γ, the mean vortex radius a, and the mean of the absolute value of the vorticity extrema ζ are shown. The solid straight-line segments show the predicted slopes based on the choice $\xi = 0.75$, which is determined from the data for ρ^{-1}. The numerical factors for the data are chosen for display purposes only.

The transformation event is the merger of the two like-sign vortices into a single new vortex. We restrict the vortex populations to those where all vortices initially have identical $\zeta_{\rm ext}$. Our transformation rules are determined by local conservation of $\zeta_{\rm ext}$ and $\mathcal{E}_s$ in its scaling form (3); since these quantities are also conserved by the Hamiltonian dynamics, they are thus global invariants for the model. However, other quantities such as ρ and Z_n are not conserved during transformations, and even $\mathcal{E}_c$ is conserved only in a scaling approximation (i.e., during any particular transformation its numerical value changes, but, insofar as the model exhibits scaling behavior consistent with our scaling theory, $\mathcal{E}_c$ is approximately conserved over many transformations).

Thus, when vortices 1 and 2 merge into vortex 3,

$$\zeta_{\rm ext3} = \zeta_{\rm ext1} = \zeta_{\rm ext2}, \quad a_3^4 = a_1^4 + a_2^4 . \qquad (5)$$

The position of the new vortex is taken as the midpoint of the line joining the original two centers. (Refinements of these transformation rules will come from further examination of the microphysics of vortex merger.)

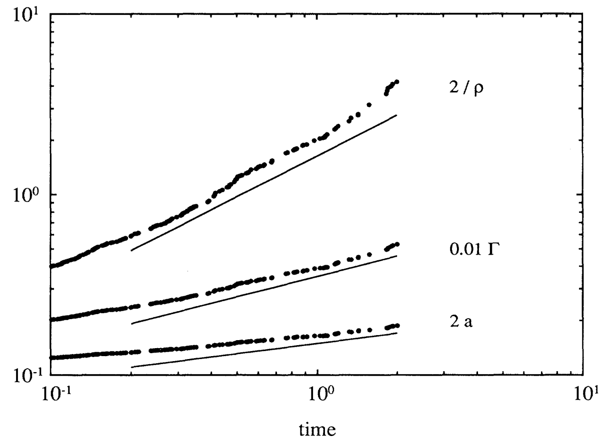

FIG. 2. Data from the modified point-vortex model. The format is as in Fig. 1. The value of mean vorticity extrema is not displayed here since it is constant in this model.

Our particular solutions are for initial conditions with 600 randomly positioned vortices with equal radius and equal numbers of each sign. Figure 2 shows statistics averaged over three such solutions. After a period of adjustment, the data compare favorably with both the turbulence solutions and the scaling theory. In particular, this demonstrates that the statistics of two-dimensional turbulence can be captured in even a simple form of deterministic, punctuated-Hamiltonian dynamics.[16]

The final point is the value of ξ. Our scaling theory does not predict this constant. However, it is determined by the frequency of close approaches among vortices, which is controlled by the chaotic Hamiltonian dynamics of point vortices; we are currently investigating the statistical mechanics of the latter. The few high-resolution turbulence and point-vortex model solutions examined so far suggest that ξ may be universal, but this evidence is still too meager to draw a firm conclusion.

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[16]While preparing the final version of this manuscript, the authors received a preprint by R. Benzi, M. Briscolini, M. Colella, and P. Santangelo, "A Simple Point Vortex Model for Two-Dimensional Decaying Turbulence" (to be published), which discusses a similar model of the vortex system.