Spontaneous Hexagon Formation in a Nonlinear Optical Medium with Feedback Mirror

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We present two-dimensional numerical simulations of a nonlinear optical system made of a thin slice of Kerr material and a feedback mirror. The phase modulation induced on the light by the nonlinear material is transformed into amplitude modulation by propagation to the mirror and back, thus forming a feedback loop. Our simulations show that the uniform plane-wave solution deforms for sufficient pump intensity into a nonuniform pattern of hexagonal symmetry, independently of the sign of the nonlinearity, a feature which may be generic for third-order nonlinear optical systems.

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Many research groups have focused their attention on models where the spatial profile of the electromagnetic field is taken into consideration and the plane-wave approximation removed. A review and extensive bibliography of such studies has recently been published as part of a special journal issue on transverse effects in nonlinear optics.[1]

Here we discuss transverse effects in a rather simple system which ought to be capable of physical realization. Our numerical simulations in two transverse dimensions seem to indicate that it has a very complex dynamics.

The system is a thin slice of Kerr medium irradiated from one side by a spatially smooth beam with a feedback mirror a distance d away to generate a counterpropagating beam in the Kerr slice, as shown in Fig. 1. In this respect it somewhat resembles the Tucson feedback-mirror experiment,[2] and indeed this experiment was a motivation for the model. A further reason lay in the counterpropagation instabilities which our group has investigated numerically:[3] The description of both diffraction and nonlinearity in the same medium leads to computationally demanding models, so that it is of interest to examine systems in which these effects are separated, to better understand their respective roles.

One further motivation leads to perhaps the most physical reason for studying such models. One does not intuitively expect spatial structures to arise in defocusing media, yet in the experiments of Giusfredi *et al.*[2] and, less directly, in those of Akhmanov, Vorontsov, and Ivanov[4] spatial structures do appear in defocusing media. The answer lies not in the linear medium itself but in the free-space propagation. What happens, at its simplest in the present model, is that a smooth beam traversing a slice of Kerr medium (which gets no spatial structure of any kind from self-action) can be phase modulated by an amplitude-modulated counterpropagating beam. A closed positive-feedback loop is possible provided that the phase-modulated transmitted input field returns from the mirror as an amplitude-modulated beam: This is precisely what propagation does.

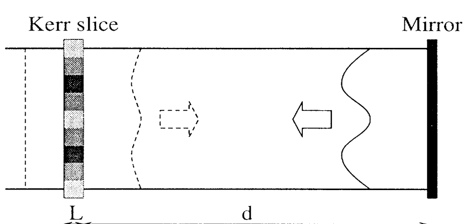

FIG. 1. Schematic diagram of the single-slice single-mirror model. The fluctuations in the carrier density modulate the phase of the field (dashed line) and diffraction changes this into amplitude modulation (solid line).

Consider a plane wave on which a small spatial phase modulation has been imposed, for simplicity a simple harmonic modulation:

$$E(\mathbf{r}) = E_0[1 + i\varepsilon\cos(\mathbf{K}\cdot\mathbf{r})]\,. \tag{1}$$

The i encapsulates the $\pi/2$ phase shift between carrier and signal characteristic of phase modulation. As this field propagates, however, the fact that the modulation has a finite transverse wave vector $\mathbf{K}$ leads to a phase slippage relative to the carrier, at a rate $K^2/2k_0$ per unit distance, where k_0 is the free-space optical wave vector. This introduces an amplitude-modulation element, and indeed for a slippage of any odd multiple of $\pi/2$, the phase modulation is completely converted to amplitude modulation. Such behavior is just what is required for the feedback loop needed for pattern formation. We further see that there is no essential difference between a phase advance (as in a self-focusing Kerr medium) and retardation (as in a defocusing medium), so that pattern formation is to be expected in both cases. Furthermore, the rather more subtle requirement for an index grating in a self-defocusing medium[3] does not apply, and Kerr media with poor spatial resolution due to diffusion or other causes can still give pattern formation.

The main difference between focusing and defocusing media is due to the change of sign of ε which introduces an additional π phase shift; so while for a focusing medium you need a $\pi/2$ phase shift for the return field to be in phase with the modulation (so that the pattern is stationary), for a defocusing medium you need a $3\pi/2$ phase shift. A $\pi/2$ phase shift would, in this latter case, induce an oscillatory dynamics with a period of two round-trip times; in the simulations presented below, however, we consider a rather sluggish medium, which cannot respond or oscillate on the round-trip time scale, so that static or slowly evolving patterns dominate. The distinction between focusing and defocusing media in this case is then primarily that patterns in the latter have higher thresholds and smaller transverse dimensions.

All these arguments are of a rather general nature and so should be applicable to many third-order nonlinear optical systems. We have tested them in a simple numerical model for the single-slice single-mirror system. Consider a Kerr medium of thickness L small enough that light transverses it in negligible time undergoing negligible diffraction. The Kerr effect is assumed to be due to a photoexcitation, of density n, which relaxes to zero with a time constant τ and has a diffusion length l_D much greater than the optical wavelength $\lambda = 2\pi/k_0$. All these hypotheses allow us to represent the slice-mirror model as[5]

$$\begin{aligned} &\frac{\partial F}{\partial z} = i\chi n F , \\ &\frac{\partial B}{\partial z} = -i\chi n B , \\ &-l_D^2 \nabla_\perp^2 n + \tau \frac{\partial n}{\partial t} + n = |F|^2 + |B|^2 , \end{aligned} \tag{2}$$

where χ parametrizes the Kerr effect (positive for a focusing medium, negative otherwise) and F and B are the forward and backward fields, respectively.

Since we neglect diffraction in the Kerr medium in order to simplify the model, we must ensure that this can be justified. We assume that diffusion washes out the wavelength-scale index grating, so that the thresholds for the feedback-mirror instabilities can be lower than for the counterpropagation instability,[4] which if above threshold would obviously invalidate the neglect of diffraction in the medium. Further, because the counterpropagation instability involves transverse wavelength scales shorter in the ratio $(d/L)^{1/2}$ than the feedback instability, it is more strongly suppressed by the transverse diffusion of n.

The detailed modulation stability analysis of this system has been presented elsewhere.[5] For the values of the parameters we have chosen the instability is of static type and the threshold is given by

$$|\chi| IL = \frac{1+\vartheta/\sigma}{2R|\sin(\vartheta)|} , \tag{3}$$

where $I \equiv |F|^2$, $\vartheta \equiv K^2 d/k_0$ is the diffraction parameter, and $\sigma \equiv d l_D^2/k_0$ parametrizes the diffusion in the slice.

We now give an overview of the dynamics of this model. True spatial pattern formation requires two transverse dimensions, which even in this simple system imposes a heavy computational demand. Our scheme simulates plane-wave excitation by using equal input fields on elements of a Cartesian grid; we have used various grid sizes from 128×128 (most of the simulations) to 512×512 (as a check). At the beginning of each simulation the equilibrium value of the carrier density is perturbed by random noise of small amplitude. The code uses a Fourier-transform routine to propagate the field in free space and a hopscotch method[6] to integrate the equation for the carrier density. With this scheme we were able to run simulations including transverse diffusion and material dynamics on a Sun SPARC station 1. In all simulations the wave-vector spectra of the patterns were monitored, and in all cases discussed it was well contained within the acceptable bandwidth.

Typically, starting close to the plane-wave solution a little above the instability threshold (i.e., with $p \simeq 1$, where p is the input intensity normalized to the threshold value), one finds spontaneous emergence of spatial modulation of the feedback field intensity. Figure 2(a) shows a typical case for a defocusing medium. In the first frame, the pattern is already fully two dimensional, but with a vaguely square symmetry. This is fairly common, and probably represents the effect of the computational grid. It is already apparent that there are dislocations or other imperfections in this square pattern, and as time evolves these grow, and the pattern distorts and reorganizes itself into the regular hexagonal pattern of the last frame, which appears to be stable. Hexagonal patterns also occur for self-focusing media, as shown in Fig. 2(b). In each case the magnitude of the dominant transverse wave vector is close to K_t, that with the lowest threshold as given by Eq. (3).

Why do hexagons form and do they coexist with the flat solution in its stable domain? To answer this question we have performed nonlinear analysis up to third order in the perturbation amplitudes for one, two and three distinct transverse wave vectors. A single vector $\mathbf{K}_1$, whose magnitude is K_t, shows the expected pitchfork bifurcation at $p=1$. This "ripple pattern" is *unstable* against growth of any second vector $\mathbf{K}_2$ of magnitude K_t. Further, for a $\pi/3$ angle between $\mathbf{K}_1$ and $\mathbf{K}_2$, $\mathbf{K}_3 = \mathbf{K}_1 - \mathbf{K}_2$ also has magnitude K_t, and is thus resonantly excited. Such modes lead to hexagonal patterns, and we find that the amplitude z of a hexagonal pattern obeys (to order z^2)

$$\begin{aligned} p = 1 &- \tfrac{1}{2} z \tan(\tfrac{1}{2}\vartheta) + \tfrac{1}{4} z^2 \tan^2(\tfrac{1}{2}\vartheta) \\ &- \tfrac{1}{2} z^2 [\cos^2(\vartheta) + 2\cos(\vartheta) - 3] . \end{aligned} \tag{4}$$

The key term is the one linear in z, wholly due to the triad coupling. It gives $z(p)$ a finite slope at $(p,z) = (1,0)$, which means that the hexagonal pattern coexists with the flat solution, appearing at $p=1$ and collapsing at $p<1$ as a "first-order phase transition." For

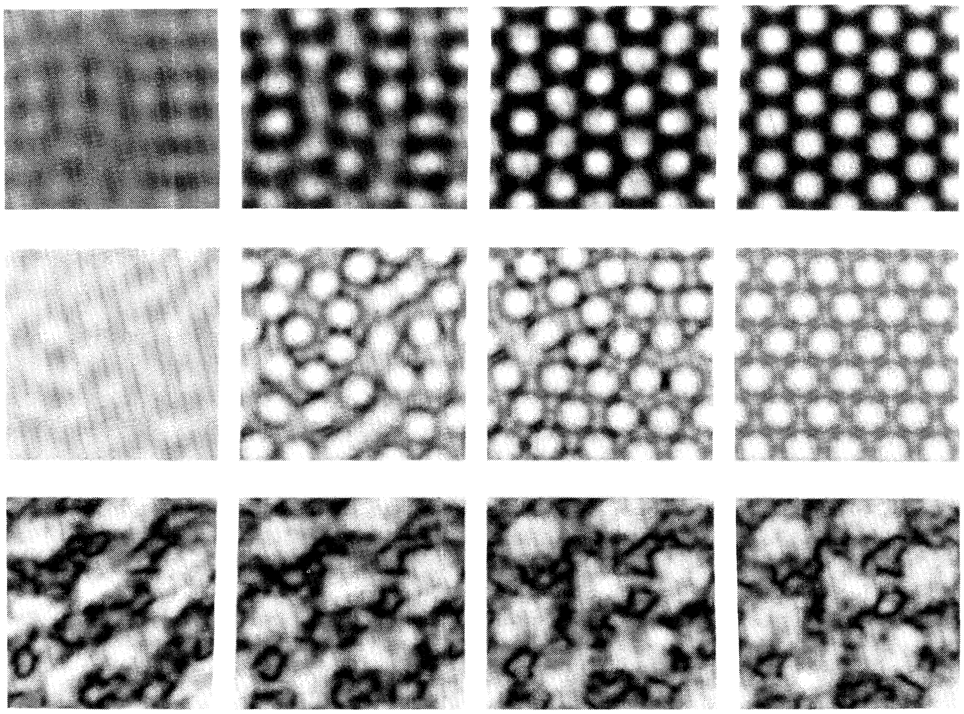

FIG. 2. Feedback field intensity patterns for the model of Fig. 1; gray-scale image, white is high intensity. In each case the modulus squared of the mirror reflectivity is $R=0.9$ and the scaled return trip time is $2dc/\tau=0.05$. (a) Hexagon formation: defocusing medium, $\sigma=0.5$, $p=1.05$. (b) Hexagon formation: focusing medium, $\sigma=1.0$, $p=1.7$. (c) "Turbulence:" focusing medium, $\sigma=10.0$, $p=2.3$.

$\vartheta=1.4839$ (corresponding to $\chi L=1$ and $\sigma=10$) this collapse is from $z\simeq0.14$ at $p\simeq0.97$. Our numerical results for these parameters are consistent with this scenario and indeed agree with (4) within about 20% of the values in z, which is very satisfactory in view of the fact that on the stable branch of (4) $z\simeq0.2$ and thus can hardly be considered small.

If the input field is increased, at roughly twice the threshold value the hexagons "melt," often forming apparently turbulent patterns, as in Fig. 2(c). We have not yet been able to identify organizing structures for this system, such as the quasisolitons in the ring-cavity case,[7] or the vortices found in laser simulations.[8,9] Bright spots seem to be born and die without any obvious underlying conservation laws. An important point about this complex dynamics is that it seems to be on a time scale related to the response time of the medium rather than optical propagation times. We do not, therefore, expect this "turbulence" to be suppressed even in rather slow media.

These last pictures suggest that this rather simple nonlinear optical system is capable of yielding amazingly complex space dynamics. Experimental realization requires strong Kerr or Kerr-like media. The beautiful and complex patterns observed by Arecchi *et al.*[10] suggest that photorefractive materials may be suitable, in particular by showing that available power levels are sufficient to generate rather complex patterns. Liquid crystals, InSb, and Na vapor are other promising candidate media.

Finally, however, we must stress that investigation of spontaneous spatial patterns in optics presents certain problems. On the experimental side there is the need for adequate power, preferably continuous wave, over relatively large areas. This indicates a use of strong, and therefore usually slow, nonlinearities involving significant and maybe complex excitation dynamics. On the theoretical side, the problem is one of a lack of analytical tools, leading inevitably to numerical simulations which in this field are extremely resource hungry. To control computer demand, one must try to capture essentials within the simplest adequate model. Such strategies include the conception and investigation of the simplest possible physical systems, of which we believe the Kerr slice with feedback mirror described here to be an example.

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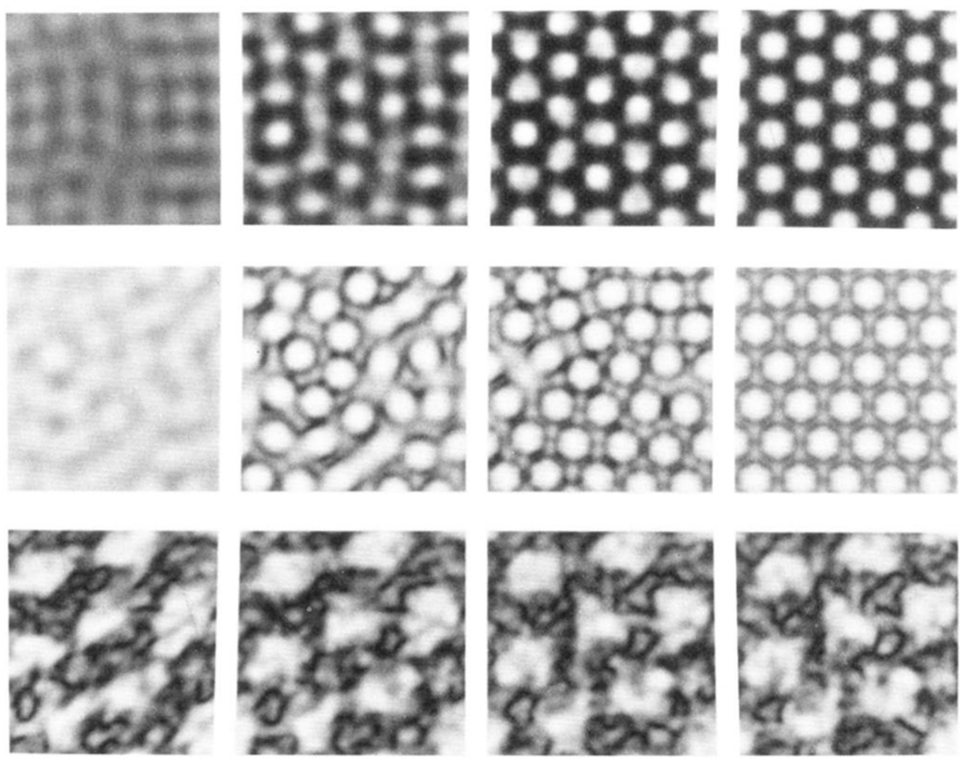

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