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Taming Chaotic Dynamics with Weak Periodic Perturbations

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The possibility of eliminating chaos in a dynamical system or of decreasing the leading Liapunov exponent by applying a *weak* periodic external forcing to the system is demonstrated through the example of a periodically driven pendulum. The application of the external forcing also results in other striking changes in the dynamics such as a stabilization of narrow subharmonic steps and the achievement of very low winding numbers.

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The inherent irregularity of chaotic dynamics and its strong sensitivity to perturbations may lead one to believe that such dynamics cannot be destroyed by means of *weak* external forcing. Moreover, the notion that the existence of three incommensurate frequencies in a system can generically lead to chaos[1] hardly suggests that the addition of an externally produced frequency will have a taming effect on chaotic dynamics. It is therefore somewhat surprising that chaos can be suppressed by external forcing. On the other hand, if such a suppression can be achieved it is highly desirable in systems of practical importance such as lasers and electronic systems. Another example is that of particle accelerators. Moreover, one can learn significant information from the response of a chaotic system to external forcing. It has been shown[2-4] that near bifurcation points of dynamical systems (e.g., period-doubling bifurcations) the application of external periodic forcing at some resonant frequencies can cause an amplification of the periodic signal and a shift in the bifurcation point, thus stabilizing the periodic state. What about the response of a system in a *deep chaotic* state? It turns out that here as well an external force may have a drastic influence. For example, it has been observed[4] that resonant parametric perturbations (on the example of the Duffing-Holmes equation) can suppress chaotic behavior. Other examples are found in Refs. 5–7. It is possible to argue that in very narrow neighborhoods of some resonant frequencies a chaotic system may yield to external parametric forcing. In the case considered in Ref. 4, the small parametric forcing is not necessarily a small perturbation on the system [in the Duffing-Holmes equation the perturbing term is proportional to $\cos(wt)x^3$, where x is a dynamical variable, and the multiplicative nature of the perturbation may render the overall perturbation relatively large]. Another means of controlling chaos has been recently suggested in Ref. 8. While the latter method seems to be efficient, it is based on a feedback mechanism: One perturbs the system in a way that is related to its position in phase space. In practical terms (applicability to various practical devices and machines is mentioned by several authors[4,8]) one needs a fast responding feedback system that produces an external force in response to the system's dynamics. While we do not presume to judge the practicality of the interesting idea suggested in Ref. 8, it seems worthwhile to explore a much simpler possibility, which is presented below.

We wish to find out whether a weak sinusoidal force can eliminate chaos in a dynamical system. To this end we choose to investigate the dynamics of a damped pendulum driven by ac and dc forces of "order 1" and a second weak ac force. This model has been studied by a

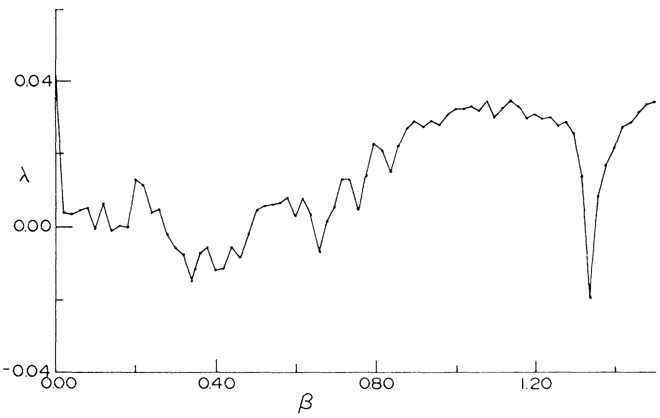

FIG. 1. Leading Liapunov exponent λ as a function of the parameter β (see text) with $G=0.7$, $A=0.4$, $I=0.905$, $w=2\pi/25.12$, and $\alpha=0.0125$. The points represent actual results and they are connected by lines to guide the eye.

large number of investigators,[9] partly because of its relative simplicity and partly because it serves as a good model for the dynamics of a Josephson junction.[9-12] The equation considered is

$$\ddot{\Theta}+G\dot{\Theta}+\sin\Theta=I+A\sin(wt)+\alpha\sin(\beta wt)\,. \qquad (1)$$

We take $G=0.7$, $A=0.4$, and $w=0.25$ [parameters for which Eq. (1) with $\alpha=0$ has been investigated before[11,12]] and study numerically the resulting dynamics as a function of the parameters α, β, and I.

When $\alpha=0$ and $I=0.905$ the solution of Eq. (1) is chaotic[11] with a leading Liapunov exponent equal to about 0.04. Consider the effect produced by adding a small perturbation with $\alpha=0.0125$. The leading Liapunov exponent λ versus the value of β is presented in Fig. 1. One observes a significant reduction in the value of λ once a nonzero value of β is switched on, as well as several ranges of values of β for which λ is negative (one of these ranges is relatively broad, $0.18\leq\beta\leq0.6$). A typical spectrum of the "voltage" (in the Josephson-junction analogy) $\dot{\Theta}(t)$ for $\beta=0.338\ldots$ is presented in Fig. 2(a). The corresponding Poincaré section is presented in Fig. 2(b), where $\Theta_n\equiv\Theta(2\pi n/w)$ and $\dot{\Theta}_n\equiv\dot{\Theta}(2\pi n/w)$.

It is interesting to consider the I-V characteristics of Eq. (1), regarded as a model of the Josephson-junction dynamics (the average voltage or winding number, V, is the time average of $\dot{\Theta}$). As is known,[10,11] when $\alpha=0$, the I-V characteristics of the system (with $G=0.7$, $A=0.4$, $w=0.25$) are composed of a set of discrete steps for which $V=wn$, n being an integer. There are narrow gaps (in I) between the neighboring steps. When a small perturbation is applied, these gaps widen and fill up with stable states (Fig. 3 for $\beta=1.118$). The nature of the in-gap dynamics can be elucidated on the basis of Figs. 4(a) and 4(b). Figure 4(a) presents the phase Θ as

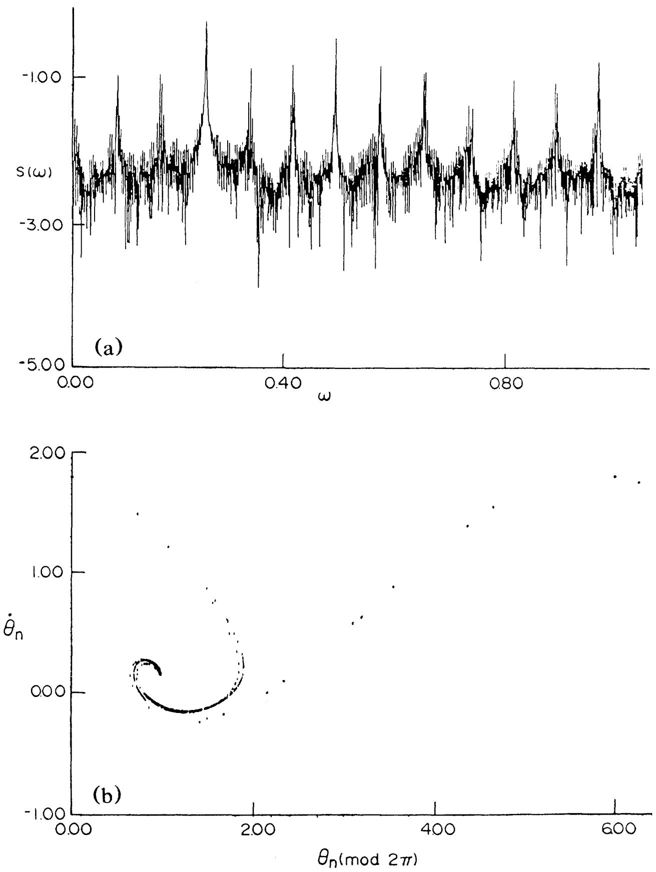

FIG. 2. (a) Power spectrum $S(w)$ of the voltage fluctuations. The parameters are $G=0.7$, $A=0.4$, $I=0.905$, $w=2\pi/25.12$, $\alpha=0.0125$, and $\beta=0.33803$. (b) Poincaré section corresponding to the system described in (a).

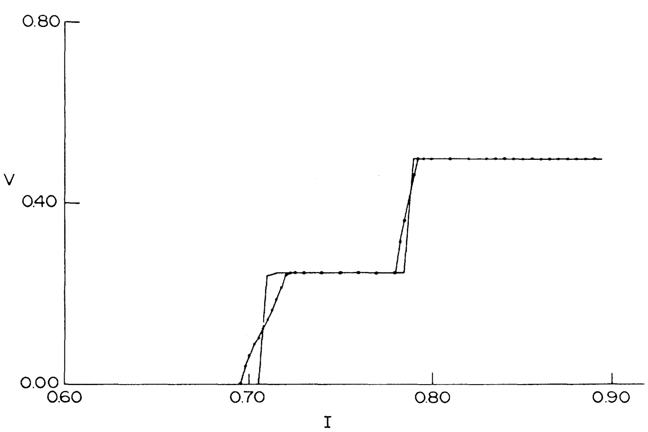

FIG 3. I-V characteristics corresponding to the parameters $G=0.7$, $A=0.4$, $w=2\pi/25.12$, $\alpha=0.0125$, and $\beta=1.11803$. The smooth curve corresponds to $\alpha=0$, i.e., no external perturbation. The dotted curve corresponds to $\alpha=0.0125$ and $\beta=1.11803$.

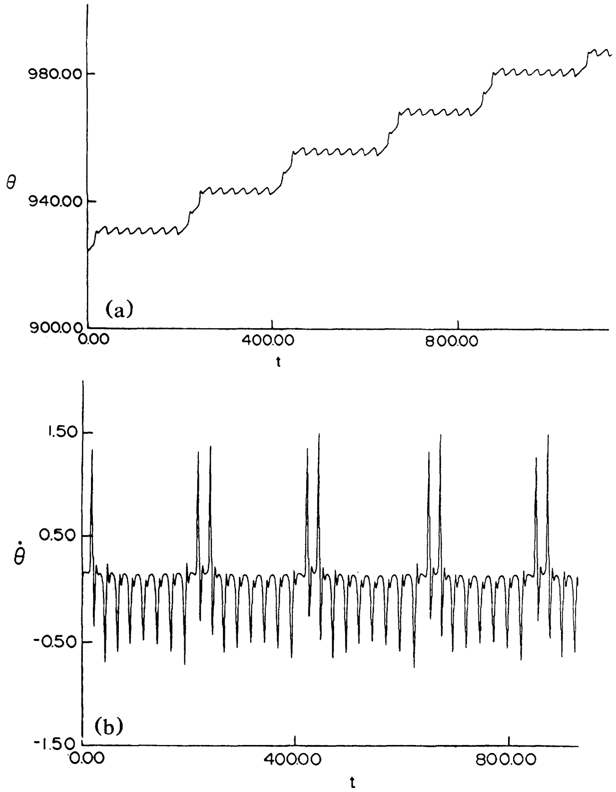

FIG. 4 (a) The phase Θ as a function of time. The parameters are $G=0.7$, $I=0.7$, $A=0.4$, $w=2\pi/25.12$, $\alpha=0.01$, and $\beta=1.11803$. (b) Instantaneous "voltage" $\dot{\Theta}$ as a function of time. The parameters are as in (a).

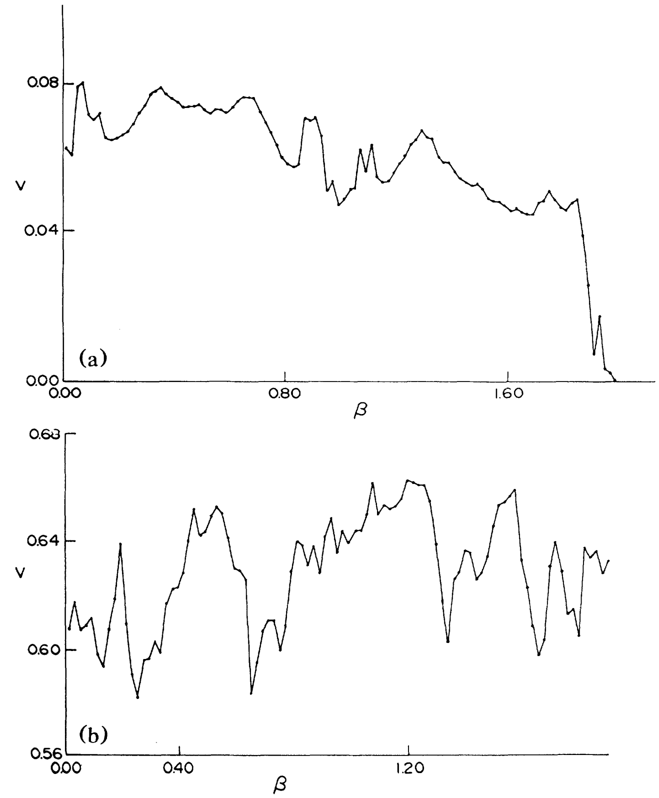

FIG. 5. (a) Dependence of the winding number ("voltage") V on the parameter β. The other parameters are $G=0.7$, $A=0.4$, $I=0.7$, $w=2\pi/25.12$, and $\alpha=0.0125$. (b) The same as (a), but $I=0.905$.

a function of time. The dynamics consists of a finite number of oscillations for which the average voltage is zero followed by a "jump" of value $2\pi/w$ and a similar set of oscillations. In Fig. 4(b) the value of the instantaneous voltage $\dot{\Theta}$ is plotted as a function of time. While a short time signal here looks like the well-known intermittent signal observed[12] for some in-gap values of the current I, it is actually fully periodic, and the system "hops" in a regular fashion between a $V=0$ "state" and a $V=0.25$ "state" (i.e., $V=w$; states refer to the $\alpha=0$ case), i.e., between the first two steps. The dependence of the winding number (average voltage) on β is demonstrated in Figs. 5(a) and 5(b). It is nonmonotonic. Some of the states corresponding to Fig. 5 are periodic and others are chaotic. Notice the very low values of voltage obtained this way.[13] The dependence of V on the value of α for parameters ($I=0.75$, $A=0.4$, $w=0.25$, $G=0.7$, $\beta=1.118$) corresponding to an $n=1$ step is depicted in Fig. 6. Notice the significant decrease in the voltage following the step ending at $\alpha=0.06$.

These results demonstrate how a relatively *weak* harmonic perturbation can change the dynamics of the system considered in a drastic fashion—chaos can be eliminated, low voltage values (and steps) can be obtained, and nonmonotonous behavior of the winding number versus the control parameters can emerge. The known sensitivity of the chaotic dynamics to perturbations of initial conditions is thus only one aspect of the sensitivity of such systems, another facet being, as demonstrated here, their response to weak, time-dependent pertur-

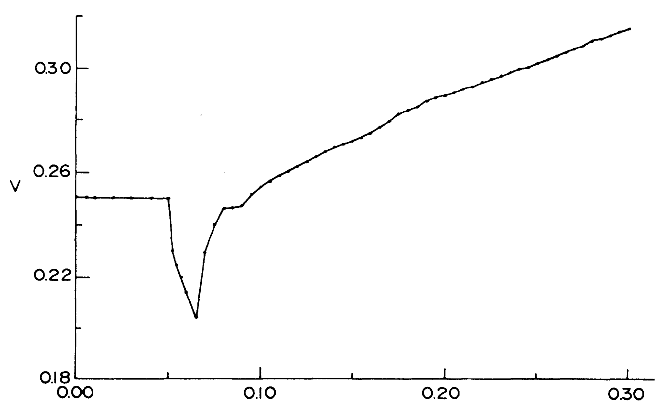

FIG. 6. α-V characteristics near step 1 (average voltage equals w). The parameters are $G=0.7$, $I=0.75$, $A=0.4$, $w=2\pi/25.12$, and $\beta=1.118033989$.

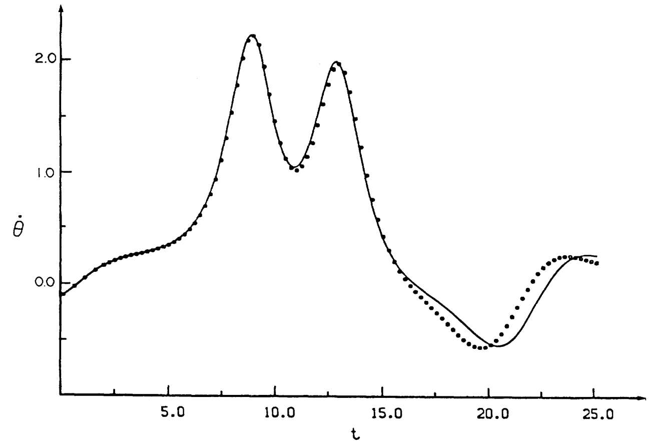

FIG. 7. Instantaneous voltage $\dot{\Theta}$ as a function of time for $0 \le t \le T$ ($T=25.12$). The parameters are as in Fig. 2(a). The smooth curve corresponds to $\alpha=0$, i.e., no external perturbation. The circles correspond to $\alpha=0.0125$. Clearly, the two solutions will depart at longer times. Both curves correspond to the same initial condition $(\Theta(t=0),\dot{\Theta}(t=0))$.

bations. A possible mechanism explaining this phenomenon is that one of the infinite unstable limit cycles embedded in the chaotic attractor[14] may be stabilized by the application of an external force. In Ref. 8 a procedure was suggested to stabilize a *given* periodic cycle. Here we demonstrate that the system has the ability to find an appropriate cycle, for a given external forcing. Figure 7 shows a plot of $\dot{\Theta}$ versus time for the unperturbed system ($\alpha=0$) compared with $\dot{\Theta}$ versus time for the perturbed one. The closeness of the two graphs is highly suggestive. For some values of β (or of the perturbing frequency) the leading Liapunov exponent is still positive, though reduced in value. We have checked the possibility that the initial conditions we have used were not in the basin of attraction of the limit cycle to be stabilized, by using a variety of different initial conditions. The resulting asymptotic state seems to be the same for all initial conditions we have used and as a result the corresponding (positive) Liapunov exponent does not change as well. The effect of a weak oscillatory perturbation on an unstable limit cycle can be modeled by the following recursion relation:

$$x_{n+1}=(\lambda+\epsilon f_n)x_n\,, \quad (2)$$

where $\lambda>1$, $\langle f_n\rangle=0$, $\langle f_n^2\rangle=1$ (e.g., $f_n=\sqrt{2}\cos n$), and angular brackets denote the average over n. When $\epsilon=0$, the fixed point x is clearly unstable, whereas for finite ϵ the Liapunov exponent η, corresponding to recursion relation (2), is

$$\eta=\mathrm{Re}\langle\ln(\lambda+\epsilon f_n)\rangle\,. \quad (3)$$

For small ϵ, $\eta=\ln\lambda-\epsilon^2/\lambda^2+O(\epsilon^3)$. Thus, when $\lambda^2\ln\lambda<\epsilon^2$ the Liapunov exponent η is negative (when $\lambda=1+\delta$, $\delta\ll 1$, this condition reduces to $|\delta|<\frac{1}{2}\epsilon^2$), i.e., x is stable. Notice that even when $\epsilon^2<\lambda^2\ln\lambda$, the forcing has an effect of reducing the Liapunov exponent, as observed in the numerical experiments. Thus, weakly unstable periodic solutions may be stabilized by oscillatory perturbations. Resonant interactions of the kind discussed in the literature may further affect the stability of these cycles.

In summary, we have shown a way to reduce chaos (i.e., the Liapunov exponent) or eliminate it altogether even in deep chaotic states. We have found that in parallel with this reduction one may stabilize narrow (and high) subharmonic steps and produce solutions with very low values of $\langle\dot{\Theta}\rangle$. Finally, we have demonstrated, using a simple model recursion relation, how a parametric resonance[15] mechanism can be responsible for this effect. Besides the practical implications of our results, we believe they may be useful in efforts to ellucidate the nature of the chaotic state itself.

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[13]Notice that the external frequency is 0.25. Obtaining a value of Θ which is very low is basically impossible for $\alpha=0$, since it corresponds to an extremely narrow step.

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