Roughness of Crack Interfaces

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The width of the crack interface in the fuse model after breakdown, w, scales with the size of the network, L, as $w \sim L^{\zeta}$. When the disorder is narrow, or when it includes arbitrarily small threshold values, we find that $\zeta = 0.7$ to within 10%, indicative of this being a universal value. This is not far from $\frac{2}{3}$, suggested by an analogy with the random directed polymer problem. When, on the other hand, the disorder is strong and includes arbitrarily large threshold values, the exponent ζ depends on the disorder. These results suggest that the random polymer problem may be relevant for brittle fracture in real materials.

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The understanding of the dynamics of growing interfaces has undergone a rapid development since the introduction of the Kardar-Parisi-Zhang (KPZ) equation, and the connection was made with the problem of directed polymers in a random medium.[1] The wide interest that has been spurred from this work has its origin in other seemingly unrelated problems, such as roughening of pinned domain walls,[2] spin glasses,[3] and wave propagation in a random medium,[4] that have been shown to be governed by the KPZ equation, thus showing the same scaling behavior as the random directed polymer.

Fracture has been studied within a statistical physics framework for some time and several interesting results concerning the scaling of various quantities describing the fracture process have been found.[5] In particular, it has been suggested that towards the end of the breakdown process, the stress field shows critical behavior.[6] However, no deep understanding of the final parts of the breakdown process exists. A natural question that arises when looking at the list of problems falling within the realm of the KPZ equation is whether the development of cracks may also be governed by it. In this Letter, we use the fuse model for brittle fracture[7] to investigate this question. It has earlier been shown that an elastic-perfect-plastic medium (or rather an electrical equivalent thereof) under strain behaves as a directed polymer in a random medium.[8] We recapitulate this argument briefly below. We then present a numerical study of the fracture surface appearing at breakdown in the fracture model. We find that the KPZ result falls within our error bars.

The random directed polymer problem may be described in the following way. We imagine a square lattice oriented at 45° with respect to two borders a distance $L/\sqrt{2}$ apart. On this lattice, we now consider all directed paths between the upper and lower borders. Each path corresponds to a given configuration of a polymer. To each bond i in the lattice we assign a random number e_i. We think of these random numbers as local energies associated with the interactions between the polymer and the medium in which it is embedded. The total energy of the polymer along a path $\mathcal{P}$ is

$$E_{\mathcal{P}} = \sum_{i \in \mathcal{P}} e_i \,. \tag{1}$$

At zero temperature, the polymer chooses the path resulting in minimum energy,

$$E = \min_{\mathcal{P}} E_{\mathcal{P}} = \min_{\mathcal{P}} \sum_{i \in \mathcal{P}} e_i \,. \tag{2}$$

The KPZ equation is a Langevin equation for this minimal path. Using a dynamical renormalization-group technique, Kardar, Parisi, and Zhang[1] found that the width of the path, defined as $w^2 = \langle x_i^2 \rangle - \langle x_i \rangle^2$, where x_i is the coordinate parallel to the borders of bond i belonging to the minimal-energy path, scales as

$$w \sim L^{\zeta}, \tag{3}$$

where the roughness exponent $\zeta = \frac{2}{3}$ in two dimensions. The value of ζ in other dimensions for the random polymer problem is still controversial.[9]

In the fuse model and the related elastic-perfect-plastic model, we also imagine a square lattice oriented at 45° with respect to two borders a distance $L/\sqrt{2}$ apart. We assume periodic boundary conditions perpendicular to the two borders, which now act as bus bars. The bonds are electrical elements that act as Ohmic resistors if the voltage difference across them is below some threshold value t_i. In the fuse model, they turn irreversibly into insulators if the voltage difference exceeds this threshold value, as Fig. 1(a) shows. In the elastic-perfect-plastic model, the electrical elements have characteristics as shown in Fig. 1(b). These elements act as Ohmic resistors up to a threshold voltage difference after which the current they carry becomes a constant which

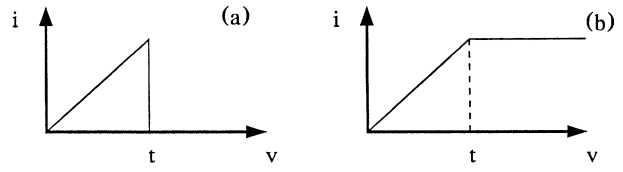

FIG. 1 (a) The voltage-current characteristics of a fuse. Here, v is the voltage difference across the fuse, and i is the corresponding current. The resistance in the linear regime is equal to 1. (b) The voltage-current characteristics of the electrical analog of an elastic–perfect-plastic element.

is independent of any further increase in voltage difference. This behavior is an electrical analog of an elastic-perfect-plastic material: Such a material will respond with a force proportional to the elongation we impose on it up to a certain threshold. If this threshold elongation is exceeded, the force will not increase.

The disorder is introduced in both the fuse model and the elastic–perfect-plastic model by assuming that the thresholds vary from bond to bond according to a spatially uncorrelated statistical distribution $P(t)$.

We now recall the connection between the elastic-perfect-plastic model and the problem of a directed polymer in a random medium.[8] Suppose that we increase the voltage difference across a network consisting of elements having characteristics as shown in Fig. 1(b). A first bond i will reach its threshold value, and as the voltage difference is further increased the current it carries will stay equal to that at t_i. Increasing the voltage difference across the network leads to more bonds reaching their threshold values. When a continuous band $\mathcal{P}$ of bonds across the network has formed parallel to the bus bars, the current flowing through the network I cannot be further increased. This maximal current must then be equal to the sum of the threshold values of each bond belonging to this band, as each threshold is equal to the maximum current the bond can carry. The band that is first to appear is the one where this sum is minimum,

$$I=\min_{\mathcal{P}} \sum_{i\in\mathcal{P}} t_i , \tag{4}$$

where $\mathcal{P}$ is a closed path around the network. From Eq. (2), we see that the problem is identical to the random directed polymer problem, with the exception that there is no *a priori* directedness imposed on the minimum-current path. However, we note that the currents flowing through the network actually impose a directedness on the paths. Thus, the problem *is* a directed one, and the width of the band, $w=(\langle y_i^2\rangle-\langle y_i\rangle^2)^{1/2}$, behaves as in Eq. (3). y_i is the coordinate perpendicular to the bus bars of bond i belonging to this band.

We now turn to the fuse model, and imagine setting up a voltage difference across the fuse network. As this potential difference is increased, fuse after fuse "blows." Before any fuse has blown, each bond carries the same current, so that the first one to go is the one with minimum threshold value. However, the currents rearrange themselves as the fuses blow, and at any stage of the breakdown process the fuse to blow next will be the one that minimizes the ratio t_i/v_i, where v_i is the voltage difference across bond i for unit voltage difference across the network. The breakdown of the network thus turns into a highly correlated process, involving the interaction of the threshold distribution with the evolving current distribution. At the final breakdown of the network, characterized by zero conductance between the bus bars, one crack has developed across the network, breaking it apart.

We note the close similarity between the elastic-perfect-plasticity model and the fuse model: The only difference between the two models is that in the behavior in the second case, the current through a bond after the threshold is reached drops irreversibly to zero rather than staying constant. The close relation between the two problems suggests that the random polymer problem indeed may be relevant for the fracture problem.

Another crude argument to justify an analogy between fracture in the fuse model and the random polymer problem is to note that when there is large disorder, the first stages of fracture consist of an uncorrelated fracture of bonds anywhere in the lattice, in the order of their strength. At the end of the process, the fracture process is highly correlated. To model this last stage, it is appealing to imagine that the final crack will minimize the number of fuses to be blown, using the holes generated in the initial stage. This leads to the final crack having the conformation of a direct polymer in a random medium with an energy distribution $e=0$ or 1 according to the state of the fuses.

There are three distinct classes of threshold distributions $P(t)$ that are relevant for the asymptotic behavior of the fracture process in the fuse model:[10] (1) distributions that behave as $P(t)=at^{\beta}$ for $t\ll 1$, (2) distributions that behave as $P(t)=1-bt^{-\beta'}$ for $t\gg 1$, and (3) those that do not have power-law tails towards small or large thresholds.

The analog of the first continuous band of elements in the plastic state in the fuse model is the *backbone* of the crack that broke the network apart. This backbone forms the "visible" surface of the crack separating the two pieces of the material, and is therefore experimentally accessible.[11] The width of this backbone we call w. We typically generated 2000 samples of size $L=5$, 1000 of size $L=10$, 500 of size $L=15$, 250 of size $L=20$, 150 of size $L=30$, and 75 of size $L=40$, for each threshold distribution $P(t)$. The current distribution at each stage of the fracture process was calculated by a conjugate gradient technique, using an error criterion $\epsilon\leq 10^{-10}$.[12] In Fig. 2 we show w as a function of lattice size. The threshold distributions used in this figure were (a) $P(t)=t^2$, (b) $P(t)=t^{1/2}$, both where $0<t<1$, (c) $P(t)=1-t^{-2}$, (d) $P(t)=1-e^{1-t}$, (c) and (d) having

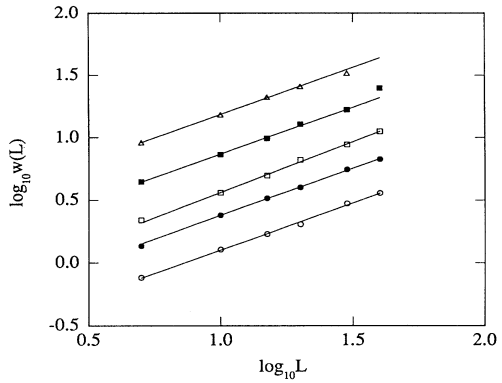

FIG. 2. The width of the final crack ignoring its branchings, w, as a function of lattice size L in the fuse model. ○, distribution (a); ●, distribution (b); □, distribution (c); ■, distribution (d); and △, distribution (e). The data have been moved apart in the vertical direction for clarity. The straight lines show the least-squares fits to the data.

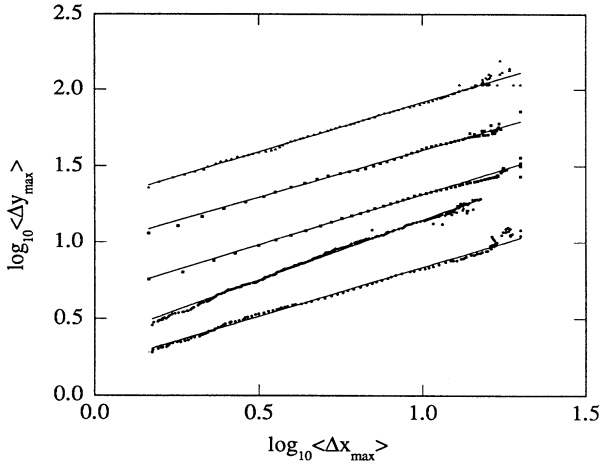

FIG. 3. $\langle \Delta y_{\max} \rangle$ plotted vs $\langle \Delta x_{\max} \rangle$. The data have been moved apart in the vertical direction for clarity, and appear in the same order as in Fig. 2. The data are based on 250 samples of size 20×20.

$1 < t < \infty$, and (e) $P(t) = (t-1)/4$, where $1 < t < 5$. Distributions (a) and (b) fall within class (1), (c) within class (2), and (d) and (e) within class (3) in the above notation. The corresponding values of ζ, defined in Eq. (3) and determined from least-squares fits, were (a) 0.76(2), (b) 0.75(2), (c) 0.81(5), (d) 0.75(5), and (e) 0.76(10). These values are typical examples of the behavior we found testing different distributions, and indicate a value for the exponent $\zeta = 0.75$ for classes (1) and (3). The data are much more noisy when the distributions belong to class (2) or (3) compared to class-(1) distributions. In addition, there seems to be a qualitative difference in the behavior when the distribution belongs to class (2), as distribution (c) does: In this case the exponent ζ seems to depend on the distribution used.

If we measure the maximum extension in both the x and y directions, $\Delta x_{\max}$ and $\Delta y_{\max}$, for the largest crack (including its branchings) throughout the *entire* fracture process, we find the following scaling relation:

$$\langle \Delta y_{\max} \rangle \sim \langle \Delta x_{\max} \rangle^{\zeta'}. \tag{5}$$

In Fig. 3 we show a plot of Eq. (5) for distributions (a)–(e). The corresponding scaling exponents ζ' are (a) 0.64(5), (b) 0.78(5), (c) 0.66(5), (d) 0.62(5), and (e) 0.65(5)—again based on least-squares fits to the data. We find it reasonable to assume that this exponent is equal to the one governing the width of the final crack, $\zeta = \zeta'$. Thus, comparing the numerical estimates we have presented for the exponents ζ and ζ', it seems likely that the data shown in Fig. 2 might overestimate the true value of the exponent, while the data shown in Fig. 3 might underestimate it. However, the internal consistency of the two sets of data is a strong indication of universality. Combining the results of these two methods, we estimate $\zeta = 0.7$ with a precision of about 10% for classes (1) and (3). The value $\frac{2}{3}$, which is the prediction for problems described by the KPZ equation, falls within the error bars of our estimate for ζ, supporting the notion of a connection between the random directed polymer problem and this problem.

Let us finally note that recently it has been shown that for certain distributions of energies e in Eq. (2) the exponent ζ may become dependent on the distribution.[13,14] The same seems to happen here for threshold distributions of class (2). Thus, the dependency of the exponent ζ on the disorder in this case is not incompatible with the possibility that this problem is in the same universality class as the random directed polymer problem. We also stress that we have done our numerical simulations on an *electrical analog* of brittle fracture. It is a possibility that the roughness exponent that we have determined here may change when models employing *elastic* elements are used.

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