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Multifractals, Operator Product Expansion, and Field Theory

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We explore possible distinctions between multifractal scaling phenomena and Lagrangian field theories (FT) describing standard critical phenomena, via the operator product expansion. While the scaling dimensions x_n of multifractal moments must be convex functions of the order n, analogous FT exponents of powers of the field are concave, by stability and correlation inequalities, and cannot describe multifractal scaling. However, powers of *gradients* of the field may lead to a novel and unexpected multifractal convexity in a FT, as, e.g., the nonlinear σ model.

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An outstanding problem[1] in statistical mechanics is to find an analytical description of complex scaling systems, like growth phenomena, critical disordered systems, and strange attractors in dynamical systems. A distinguishing feature of these complex scale-invariant phenomena is the existence of a continuous spectrum of scaling indices and a corresponding *multifractal* singularity spectrum $f(\alpha)$.[2-5] It arises from the *nonlinear* dependence on q of the exponents $\tau(q)$ of the moments of a multifractal measure $\mu(\mathbf{r})$ (or local probability of events at point $\mathbf{r}$) defined by

$$\sum_{\mathbf{r}} \mu^q(\mathbf{r}) \sim (a/R)^{\tau(q)}, \tag{1}$$

where the sum extends over the support of the measure of typical linear size R (a is a microscopic cutoff). It has been a challenge to calculate nontrivial functions $\tau(q)$ analytically. A related question[1] is whether the universal features of multifractals could be described by a (Lagrangian) field theory (FT),[6] as for critical phenomena (with conformal invariance in two dimensions[7]). Some field-theoretic methods have been applied successfully to multifractal phenomena, such as harmonic diffusion near absorbing fractals,[8] electron localization,[9] random resistor networks,[10] and dilute ferromagnets,[11] a nonexhaustive list. A characteristic of these works (except Ref. 8), however, is the use of an analytical continuation of some Lagrangian FT (replica method), which leads to unusual features, to be described below.

The aim of this Letter is to compare multifractals and FT. We first provide a formal but unifying field operator description of multifractal moments. Strikingly, multifractal correlations then follow the rules of *an operator product expansion.* However, we show that these multifractal moment operators are really distinguished from analogous *field power* operators in a standard FT, due to stability requirements in the latter. This is quantified by a general inequality on scaling dimensions, the "θ criterion," which distinguishes the multifractal from the FT operators. We finally suggest that powers of field *derivatives* in a FT could escape this stringent criterion and display multifractal behavior.

A natural way to describe multifractals in FT terms is to consider local random "events" $O(\mathbf{r})$ building up the measure $\mu(\mathbf{r})$, such that their moments $O^q(\mathbf{r})$ scale like

$$\mu(\mathbf{r}) \equiv \frac{O(\mathbf{r})}{\Sigma_{\mathbf{r}} O(\mathbf{r})}, \quad \overline{O^q(\mathbf{r})} \sim \left(\frac{a}{R}\right)^{x_q}, \tag{2}$$

where the overbar represents a space (i.e., disorder) average and where x_q is the "scaling dimension" of O^q. From (1) and (2), we get

$$\tau(q) = x_q - D - q(x_1 - D), \tag{3}$$

where D is the fractal dimension of the measure support, associated with the total mass $\sum_{\mathbf{r}} 1 \equiv M \sim (R/a)^D$, such that $D = -\tau(0)$, and $x_0 = 0$. If only one (or a few) fractal dimension is present, then all generalized[4] (or "critical"[2]) dimensions $D(q) \equiv \tau(q)/(q-1)$ reduce to $D(q) = D$ for any q. By (3) this corresponds to $x_q = q x_1$, i.e., *gap scaling*. In general, the $\tau(q)$, like x_q, are nonlinear functions of q.

It is now very instructive to consider *multifractal correlations*. In terms of events $O(\mathbf{r})$ and dimensions x_q, the scaling factorization proposed by Cates and Deutsch[12] reads

$$\overline{O^p(\mathbf{r}_1) O^q(\mathbf{r}_2)} \sim r_{12}^{\theta} R^{-x_{p+q}} + \cdots , \tag{4}$$

where $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2| \ll R$, and

$$\theta = x_{p+q} - x_p - x_q . \tag{5}$$

We restrict ourselves to $p, q \geq 0$.[13]

Consider now a given *field theory* describing the vicinity of a critical point and its *scaling operators* $\mathcal{O}_i(\mathbf{r})$ of dimensions x_i. They satisfy the operator product expansion[6,14] (OPE)

$$\mathcal{O}_i(\mathbf{r}_1)\mathcal{O}_j(\mathbf{r}_2) = \sum_k c_{ijk} r_{12}^{x_k - x_i - x_j} \mathcal{O}_k(\mathbf{r}_2) , \tag{6}$$

where c_{ijk} are universal, the sum being ordered such that x_k increases with k. By d-dimensional conformal invariance,[7] the critical three-point functions read

$$\langle \mathcal{O}_i(\mathbf{r}_1)\mathcal{O}_j(\mathbf{r}_2)\mathcal{O}_k(\mathbf{r}_3)\rangle = \frac{c_{ijk}}{r_{12}^{x_i+x_j-x_k} r_{23}^{x_j+x_k-x_i} r_{31}^{x_k+x_i-x_j}} . \tag{7}$$

In the presence of a finite large length scale R (finite size or correlation length ξ), each $\mathcal{O}_k$ gets an expectation value $\langle \mathcal{O}_k \rangle = a_k R^{-x_k}$ (a_k can vanish, e.g., for symmetry reasons). Hence for $r_{12} = r \ll R$, Eq. (6) implies *factorization* of the two length scales r, R in

$$\langle \mathcal{O}_i(\mathbf{r})\mathcal{O}_j(\mathbf{0})\rangle = \sum_k c_{ijk} a_k r^{x_k - x_i - x_j} R^{-x_k} . \tag{8}$$

If the *identity* operator $\mathcal{O}_0 = \mathbf{1}$ of lowest dimension $x_0 = 0$ is present, we recover standard gap scaling $\langle \mathcal{O}_i \mathcal{O}_j \rangle \sim r^{-x_i - x_j}$ as $r \to 0$.

Now, we remark that this OPE [Eqs. (6) and (8)] of standard critical phenomena has a direct analog in *multifractal* correlations: Associate abstract operators $\mathcal{O}_q(\mathbf{r})$ of a hypothetical field theory describing a multifractal to fluctuating moments $O^q(\mathbf{r})$ [Eq. (2)] via

$$\langle \mathcal{O}_p(\mathbf{r}_1)\mathcal{O}_q(\mathbf{r}_2) \cdots \rangle = \overline{O^p(\mathbf{r}_1) O^q(\mathbf{r}_2) \cdots} ,$$

where angular brackets denote field averages. The validity of this form for integer p, q can be verified in those multifractals[8,9,11] which have been studied by FT methods. Then the factorized multifractal correlation (4) is just the expectation value of the leading term in the OPE of the moment operators $\mathcal{O}_p, \mathcal{O}_q$,

$$\mathcal{O}_p(\mathbf{r})\mathcal{O}_q(\mathbf{0}) \sim r^{x_{p+q} - x_p - x_q} \mathcal{O}_{p+q}(\mathbf{0}) + \cdots . \tag{9}$$

Hence, multifractal moments (p,q) couple directly to $p+q$, up to subleading operators $\mathcal{O}_k$ with $x_k > x_{p+q}$. Being associated with a probability distribution, the continuous set x_q $(q > 0)$ is in general[15] determined by the discrete x_n ($n \in \mathbb{N}^*$), on which we now focus.

Field theories involve infinities of scaling operators with a nontrivial spectrum. Consider the fluctuating field variable $\varphi(\mathbf{r})$ describing the local magnetization in the d-dimensional Ising model, $\varphi(\mathbf{r}) = L^{-d} \sum_{i \in L^d} \sigma_i$, where the spins σ_i are coarse grained[16] over a box of size L, centered about $\mathbf{r}$, with $a \ll L \ll \xi (\equiv R)$. Near the critical point, the fluctuations of φ are described by an effective Lagrangian $\mathcal{L} = (\nabla\varphi)^2 + \xi^{-2}\varphi^2 + g\varphi^4$. The nth moment $\varphi^n(\mathbf{r})$ has the scaling form[16]

$$\begin{aligned} \langle \varphi^n(\mathbf{r}) \rangle_L &= (L/a)^{-n x_1} f_n(\xi/L) , \\ f_n(y) &= \sum_k c_k y^{-x_k} , \end{aligned} \tag{10}$$

where x_1 is the spin dimension ($2x_1 = d - 2 + \eta$), and the scaling function f_n is an expansion over the operators $\mathcal{O}_k$ appearing in the iterated OPE (6) of a *cluster* of n operators φ. Equation (10) exhibits *gap scaling* in the critical limit $y = \xi/L \to \infty$ (for $c_0 \neq 0$ and $x_0 = 0$), due to the presence of the identity operator $k = 0$ in the OPE (6), while subleading operators give corrections to scaling. Among these, there is the so-called *normal-ordered product (renormalized) operator* $\mathcal{O}_n(\mathbf{r}) = :\varphi^n:(\mathbf{r})$ of dimension x_n, which corresponds to the global scaling of a cluster of n spins. For instance, $:\varphi^2:$ is the subtracted energy operator $\sigma\sigma$ with $x_2 \equiv (1-\alpha)/\nu = d - 1/\nu$. x_n is a nonlinear function of n. These higher operators $\mathcal{O}_n$ (irrelevant in the renormalization-group sense for n high enough) may be, for symmetry reasons,[17] directly *observable*. In Eq. (10) this corresponds to a leading term $f_n(y) \sim c_n y^{-x_n}$ for $y \to \infty$, so that $\langle \varphi^n \rangle_L \sim L^{x_n - n x_1}$, and gap scaling breaks down. This is the case of the spin-wave operators of the XY model, $\mathcal{O}_n(\mathbf{r}) = \varphi^n(\mathbf{r}) = e^{in\theta(\mathbf{r})}$, where $\theta(\mathbf{r})$ is a 2D rotator angle. The OPE of two such $\mathcal{O}_n$, $\mathcal{O}_{n'}$ starts with $\mathcal{O}_{n+n'}$, as in the multifractal case (9). Another striking example is given by polymers (self-avoiding walks) with a star topology[18] (Fig. 1). There, a new scaling operator $\mathcal{O}_n(\mathbf{r})$ is associated with the singular core at $\mathbf{r}$ where n long polymers are chemically joined together. The probability of approach at a distance r, $P_{n,n'}(\mathbf{r})$, of two stars of n and n' arms (Fig. 1) is then given by the OPE of operators $\mathcal{O}_n(\mathbf{r})$ and $\mathcal{O}_{n'}(\mathbf{0})$. The

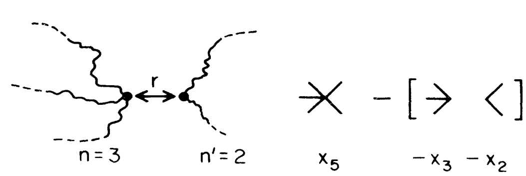

FIG. 1. Three- and two-arm stars approaching each other at r, and the subtraction rule illustrating OPE (11).

resulting *fused* object for $r \rightarrow 0$ is clearly a $n+n'$ star; hence

$$P_{n,n'}(\mathbf{r}) \sim \mathcal{O}_n(\mathbf{r})\mathcal{O}_{n'}(\mathbf{0}) \sim r^{\theta}\mathcal{O}_{n+n'}(\mathbf{0}) , \qquad (11)$$

with a *contact exponent*[18] $\theta \equiv x_{n+n'} - x_n - x_{n'}$ as in (9).

Now, because of the above formal analogies, can one not derive the (continuum) spectrum x_q [Eqs. (2) and (3)] of multifractal moments from a (denumerable) spectrum x_n of composite moment operators $\mathcal{O}_n = {:}\varphi^n{:}$ in a Lagrangian field theory? We show that an essential *distinction* actually seems to exist. It arises from the *convexity* properties of the multifractal spectrum $\tau(q)$ or x_q, which are incompatible with the *stability* requirements for a conventional FT.

Multifractals.— Exponents x_q are related to moments (2) of a probability distribution, and therefore satisfy the convexity[19] $d^2x_q/dq^2 \leq 0$. The "singularity spectrum"[5] $f(\alpha)$, the Legendre transform of $\tau(q)$, satisfies $f''(\alpha) = 1/\tau''(q) = 1/x_q'' \leq 0$; hence $f(\alpha)$ is *convex,* as required for the standard saddle-point integration[5] over the "singularity strength" α. Since $x_0 = 0$,

$$\theta = x_{p+q} - x_p - x_q \leq 0 \quad (pq \geq 0) , \qquad (12)$$

implying decay of the correlation function (4) with the distance r along the fractal support. (Notice that $\theta \geq 0$ iff $pq < 0$.)

Critical phenomena.— For a "standard" field theory, we state that the spectrum $\{x_n\}$ of field powers described above [Eqs. (10) and (11)] obeys just *reversed* inequalities. For (geometrical) critical systems[18] like self-avoiding star polymers, consider the short-distance expansion (11) which starts at level $n+n'$. The probability of approach $P_{n,n'}(r)$ of two star cores (Fig. 1) must vanish as $r \rightarrow 0$, because of the short-distance *repulsion* of the critical objects. Hence the contact exponent θ must be positive

$$\theta \equiv x_{n+n'} - x_n - x_{n'} \geq 0 , \qquad (13)$$

as opposed to (12). This generalizes directly to *geometrical* critical properties of percolation and Potts clusters, or O(n) models in their graphical representations.[18]

In the general case of composite operators like the powers φ^n of the field in, e.g., φ^4 theory, the singular contributions due to gap scaling dominate in the OPE the scaling behavior of the renormalized (cluster) operator ${:}\varphi^n{:}$, as in Eq. (10). However, a φ^4 field theory has a Symanzik (polymer) representation in terms of repulsively interacting Brownian paths.[20] Ultimately, their short-distance repulsion, e.g., that of the φ^4 potential, reflects itself in the important inequality $x_n \geq nx_1$, obeyed[21] by the scaling dimension x_n of operator ${:}\varphi^n{:}$. This convexity property, which excludes multifractal behavior as in (12), is related to the general *correlation inequalities.*[20,22] For example, in the d-dimensional Ising model, the four-spin *connected* correlation function obeys the Lebowitz inequality[20,22] $\langle \sigma_1\sigma_2\sigma_3\sigma_4\rangle_c \leq 0$, which we recast as

$$\langle 1[(23) - \langle 23\rangle]4\rangle \leq \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle , \qquad (14)$$

where $1 \equiv \sigma_1$, etc. Consider this in the limit $a = r_{23} \ll r = r_{14} \ll R = r_{12} \simeq r_{13}$ (Fig. 2). Then $\epsilon = \sigma_2\sigma_3 - \langle\sigma_2\sigma_3\rangle$ is the subtracted *energy* operator with dimension $x_\epsilon = x_2$ (corresponding to ${:}\varphi^2{:}$ in the renormalized φ^4 theory). Owing to (7) the *critical* three-point function on the left-hand side of Eq. (14), positive by Griffiths's inequality, reads

$$\langle \sigma(\mathbf{0})\epsilon(\mathbf{R})\sigma(\mathbf{r})\rangle = \frac{c_{\sigma\epsilon\sigma}}{R^{x_\epsilon}|\mathbf{R}-\mathbf{r}|^{x_\epsilon} r^{2x_\sigma - x_\epsilon}} , \qquad (15)$$

where $x_\sigma \equiv x_1$ is the spin scaling dimension. For $r \ll R$ the right-hand side of (14) is $\langle 12\rangle\langle 34\rangle \simeq \langle 13\rangle\langle 24\rangle = R^{-4x_\sigma}$. Hence the correlation inequality (14) implies for $r/R \rightarrow 0$, $x_\epsilon - 2x_\sigma \geq 0$, i.e., $x_2 \geq 2x_1$, a rigorous inequality in any d. In 2D, Onsager values are $x_1 = \eta/2 = 1/8$ and $x_2 = 2 - 1/\nu = 1$, while in 3D (Ref. 22), $x_1 \cong 0.52$, $x_2 \cong 1.41$. A similar convexity for higher-order operators could perhaps be obtained from that[23] of the Ursell functions $\langle \sigma_1 \cdots \sigma_{4n}\rangle_c \leq 0$. Again, correlation inequalities in a φ^4 FT express the *repulsive properties* of φ^4 potentials[20] similar to that of star polymers above. We now give physical examples.

Harmonic diffusion and polymers.— Cates and Witten[8] considered the Laplacian diffusion field ϕ near a fractal absorber constituted by a random walk (RW) in $d = 4 - \epsilon$ dimensions. Generalizing to the case where the only chemically adsorbing sites are the *multiple intersection points of order L* of the RW, we find a *multifractal* spectrum (3) of moments

$$\langle \mathcal{O}_n\rangle = \overline{\phi^n}$$

broadening with L,

$$\begin{aligned} x_{L,n} - x_{L,1} &= (n-1)[D_L(n) - D] \\ &= -Ln(n-1)\epsilon^2/4 + O(\epsilon^3) , \end{aligned} \qquad (16)$$

where $D = d - L(d-2)$ is the fractal dimension of the *active sites.* Equation (16) obeys the multifractal inequality (12), as opposed to the scaling dimensions x_n of the usual n-arm self-avoiding polymer stars:[18] $x_n - nx_1 = n(n-1)\epsilon/8 + O(\epsilon^2)$ [$x_n = (9n^2 - 4)/48$ in 2D], satisfying repulsive FT convexity (13).

Pure and random magnets.— The convexity of exponents of ordinary critical phenomena is observed in the

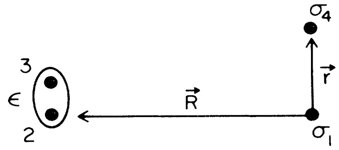

FIG. 2. Energy operator $\epsilon(\mathbf{R})$ and spins $\sigma(\mathbf{0})$, $\sigma(\mathbf{r})$.

O(N) vector model in $d=4-\epsilon$ dimensions.[24] For an nth power of the order-parameter field φ_a, $a=1,\ldots,N$, we have the scalar (renormalized) normal-ordered products $:(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^m:$ for which $x_{2m}-mx_2=6m(m-1)\varepsilon/(N+8)+\cdots$, or the traceless symmetric tensors $\varphi_1\cdots\varphi_n$ with $x_n-nx_1=n(n-1)\varepsilon/(N+8)+\cdots$. Both obey the FT convexity (13) and cannot describe multifractal moments.

On the other hand, *ferromagnetic* spin systems with *quenched disorder* [11] display multifractal (attractive) convexity. There the local random variable O (the "event") is the local magnetization (say on a lattice site) in a fixed disorder configuration $\mathcal{C}$, $O(\mathbf{r})=\langle\sigma(\mathbf{r})\rangle_{\mathcal{C}}$. Near T_c, one has scaling

$$\overline{\langle\sigma(\mathbf{r})\rangle^n}=\langle\mathcal{O}_n(\mathbf{r})\rangle\sim\xi^{-x_n},$$

where local operators $\mathcal{O}_n(r)$ are those of a replica FT.[11] A perturbative expansion about the $Q\geq 2$ random 2D Potts model gives[11] $x_n-nx_1=-yn(n-1)/16+O(y^2)$, where $0\leq y=a(Q-2)\ll 1$. Hence a random ferromagnet typically exhibits multifractal behavior,[11] i.e., inequality (12), in contradistinction to the pure Potts model. Convexity (12) expresses the known effective *attraction* between replicas.

Can one fulfill $\theta<0$ in a FT?—So far, we have seen that operators like *powers of the field* cannot describe multifractals in a stable Lagrangian FT, ultimately because of their short-distance "repulsion." However, other classes of operators may not be repulsive, especially those with *derivatives.* We illustrate these points with the O(N) nonlinear σ model in $2+\epsilon$ dimensions. Wegner[25] has recently calculated the dimensions x_{2s} of scalar *gradient* operators of the form $\mathcal{O}_{2s}=(\partial\pi_a\partial\pi_a)^s$, where π_a is the field, $a=1,\ldots,N$. He finds $x_{2s}=2s-s(s-1)\epsilon/(N-2)+O(\epsilon^2)$, and hence an *attractive* convexity $\theta<0$ for the *a priori stable* FT ($N>2$). The above result, for powers of *gradients* and not of the field itself, could provide the first example of *multifractal behavior in a Lagrangian stable field theory.* But, just because of the downward bending of x_{2s}, these operators could also become *relevant* ($x_{2s}<d$) for large s, and destroy the usual fixed point of the nonlinear σ model,[25] which would then have to be reanalyzed entirely.[25] If, on the other hand, we consider the nth *powers* of the field π_a, the leading scaling dimension $x_n=n(n+N-2)\epsilon/2(N-2)+O(\epsilon^2)$, associated with the traceless symmetric tensors,[26] still satisfies standard FT "repulsive" convexity (13) for $N>2$. When crossing $N=2$, it changes to "attractive" convexity (12). This can be directly traced back to a *change of sign* of the fixed-point coupling[26] $g^*=\epsilon/(N-2)+O(\epsilon^2)$, and thus to the instability $g^*<0$, for $N<2$. A similar (unstable) spin model describes multifractal Anderson localization.[9]

In summary, we have provided a field-theoretic formalism for multifractals and stressed the role of the OPE. For the field-power class, a θ criterion distinguishes multifractal from standard critical behavior. It can be applied to a variety of other physical systems, like (multicritical) hexatic liquid crystals,[17,27] which can be shown to be *nonmultifractal,* or to continuous moments of Ising magnetization.[28] However, classes of *derivative* operators may lead to an unexpected multifractal behavior in a (stable) FT, which is far from being understood. In particular, it would be interesting to know the spectrum of operators $(\nabla\varphi)^{2s}$ in a φ^4 FT.

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