Partition Function of Magnetic Random Walks

In a recent Letter,[1] Broda introduced a representation of scalar quantum electrodynamics (QED), coupled to a particle of charge e, in terms of (closed and open) random walks (RW) or, more precisely, Brownian paths. The latter are interacting via short-range repulsive δ distributions, as in the Edwards model of polymers,[2] but also via long-range *magnetic* inductions generated by currents flowing along the RW. This representation generalizes to QED that by Symanzik[3] of φ^4 field theory. Broda infers the triviality of (scalar) QED above $d=4$ in Euclidean d-space. The long-range intersections of random walks or Brownian paths are known (see references in Ref. 1) to disappear in $d>4$ (leading to the triviality of φ^4 theory in $d>4$). Then, a scaling analysis of the long-"time" behavior of the *magnetic* RW indicates that the renormalized magnetic partition function is also trivial in $d>4$. However, as noted by Broda, the analysis is "very qualitative," since the ultraviolet (UV) regularization is not handled and could restore interactions. In this Comment, I address this question by computing *explicitly* the first terms of the RW magnetic partition function, and using an earlier integral geometry result about mutual inductances of arbitrary closed curves or manifolds.[4] The results support the above conjecture[1] that magnetic RW develop long-range interactions only below $d=4$.

The dimensionless magnetic connected partition function of two closed Brownian paths $\mathcal{B}_1$ and $\mathcal{B}_2$ of length S is defined as[1,5] ($g=e^2$)

$$\mathcal{Z}=S^{-d/2}\int d^dx\langle e^{gM}-1\rangle_{\text{Brown}}\,, \tag{1}$$

where $\mathbf{x}$ is the relative position of their centers of mass and M their mutual inductance in d dimensions,[4]

$$M\equiv\frac{1}{S_d(d-2)}\oint_{\mathcal{B}_1\times\mathcal{B}_2}\frac{d\mathbf{r}_1\cdot d\mathbf{r}_2}{\|\mathbf{r}_1-\mathbf{r}_2\|^{d-2}} \tag{2}$$

with $S_d=2\pi^{d/2}/\Gamma(d/2)$, and $\mathbf{r}_1,\mathbf{r}_2$ the closed Brownian trajectories over which (1) is averaged. Notice that in Ref. 1 [Eq. (11)], both closed and open magnetic RW appear in QED. Their UV scaling behavior is actually the same, hence the issue of triviality of QED. We choose closed RW here for mathematical convenience.[4]

By reversing the orientation of one curve, one has trivially $\langle M\rangle\equiv 0$. Hence $\mathcal{Z}$ has the expansion $\mathcal{Z}=\frac{1}{2}g^2\int d^dx\langle M^2\rangle_{\text{Brown}}+\cdots$. For closed curves $\mathcal{C}_1$ and $\mathcal{C}_2$ of fixed but arbitrary shapes, I have considered the kinematic integral $\mathcal{I}\equiv\int d^dx\langle M^2\rangle_{\text{ang}}$ over relative rigid Euclidean motions of the two curves, $\langle\cdots\rangle_{\text{ang}}$ being an angular average. The following factorization theorem holds:[4]

$$\mathcal{I}=j_d\int_0^\infty dp\,p^{d-3}\mathcal{A}_1(p)\mathcal{A}_2(p)\,, \tag{3}$$

where $j_d=S_d(d-1)!(2\pi)^{-d}$, and where $\mathcal{A}_i$, $i=1,2$, are characteristic functions of each curve $\mathcal{C}_i$ separately, given by some explicit integral formulas.[4] Each function $\mathcal{A}_i$ in the factorized integrand of (3) now can be averaged *independently* over the Brownian configurations of $\mathcal{B}_i$. For a closed Brownian ring of length S, the average is[5]

$$\langle\mathcal{A}(p)\rangle_{\text{Brown}}=\frac{1}{d-1}\frac{2}{p}\int_{s_0}^{S}ds(S-s)(A+p^2B)e^{-cp^2}\,, \tag{4}$$

with $A=\delta(s)-1/S$, $B=-(\frac{1}{2}-s/S)^2$, $c=\frac{1}{2}s(1-s/S)$, and s_0 a *microscopic* cutoff.

Substituting (4) for both $\mathcal{A}_i$, $i=1,2$, in (3), we perform the integral over p,

$$\mathcal{Z}=g^2\frac{j_dS^{-d/2}}{(d-1)^2}\int_{s_0}^{S}ds(S-s)\int_{s_0}^{S}ds'(S-s')I+\cdots\,,$$

$$I=\sum_{n=0}^{2}a_n(c+c')^{n-d/2}\Gamma(\tfrac{1}{2}d-n)\,,$$

with $a_0=BB'$, $a_1=AB'+A'B$, $a_2=AA'$.

For $d>4$, the Γ functions in I are well defined. But UV singularities dominate the double integral as $s_0\to 0$. The *first* such divergence appears at $d=4$ in the $n=0$ term. The other apparent pole at $n=2$, $d=4$ due to $\Gamma(\frac{1}{2}d-2)$ is actually canceled since $\int_0^S ds(S-s)a_2\equiv 0$. Hence for $d<4$ the integrals are finite as $s_0\to 0$ and develop a singularity as $d\to 4^-$, $\mathcal{Z}=g^2(2\pi)^{-2}S^{4-d}/6(4-d)+\cdots$. Hence, as conjectured in Ref. 1, the infrared behavior for large paths $S\to\infty$ is nontrivial in $d<4$, scaling perturbatively like $(1/\varepsilon)S^\varepsilon$, $\varepsilon=4-d$, as in the case of polymers.[6] For $d>4$, we find a cutoff term $\mathcal{Z}=a_dg^2S^{2-d/2}s_0^{2-d/2}$, vanishing as $S\to\infty$. Q.E.D.

Bertrand Duplantier
Service de Physique Théorique de Saclay
F-91191 Gif-sur-Yvette CEDEX, France

Received 18 June 1990
PACS numbers: 12.20.Ds, 03.50.−z, 05.40.+j

[1] B. Broda, Phys. Rev. Lett. **63**, 2709 (1989).
[2] S. F. Edwards, Proc. Phys. Soc. **85**, 613 (1965).
[3] K. Symanzik, in *Local Quantum Theory,* edited by R. Jost (Academic, London, 1969).
[4] B. Duplantier, Commun. Math. Phys. **82**, 41 (1981); **85**, 221 (1982); see also, W. F. Pohl, in *Proceedings of the International Symposium in Honor of N. H. Kuiper, Utrecht, 1980,* Lecture Notes in Mathematics (Springer, Berlin, 1981).
[5] B. Duplantier (to be published).
[6] J. des Cloizeaux, J. Phys. (Paris) **42**, 635 (1981).

