Fixed Scale Transformation Approach to the Nature of Relaxation Clusters in Self-Organized Criticality

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We use the fixed scale transformation method, developed for fractal growth, to investigate analytically the nature of clusters in self-organized criticality (SOC). In two dimensions the clusters of sites involved in a relaxation process turn out to be compact ($D=2$) because of the absence of effective screening. Therefore they are more similar to Eden-type clusters (possibly with a rough surface) than to those of the usual fractal growth models. This result is in good agreement with the computer simulations and one can conjecture that it should hold for any dimension. The critical state corresponding to SOC dynamics is therefore of much simpler nature with respect to those of the usual fractal growth models.

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The class of cellular automata that go under the name of self-organized criticality[1] (SOC) has attracted a large interest within the physics community and beyond. The reason is that these models reach a critical state as a result of their own dynamics without the adjustment of external parameters. Strictly speaking, this intrinsic criticality could be already observed in the standard models of fractal growth like diffusion-limited aggregation[2] (DLA) or the dielectric breakdown model[3] (DBM) and also in other models like, for example, invasion percolation[4] (IP). The SOC-type models are, however, mathematically simpler than DLA and DBM because they do not involve a long-range coupling field. Therefore it is nontrivial to investigate what are the minimal mathematical ingredients that can give rise to an intrinsically critical dynamics.

The activity in this field consists mostly of computer simulations[1,5] with relatively few analytical approaches.[6-8] These have mainly referred to the distribution of cluster sizes that can be easily obtained by mean-field arguments. In this Letter we show that the method of the fixed scale transformation[9] (FST), developed for fractal growth models, can be successfully applied to study analytically the nature of clusters (whether they are compact or fractal) in SOC models. A cluster is defined as the set of sites involved in a relaxation process (avalanche). It is interesting to remark that such a problem cannot be addressed by the usual methods like the renormalization-group method. The result for the two-dimensional case shows that these clusters are compact at all length scales. This implies that, statistically, no empty regions of any size are left by the growth process that generates the clusters. Therefore, their fractal dimension D is equal to the Euclidean dimension d ($D=d=2$). This result is nontrivial because the model has no length scale built in. It is also possible to understand why there is no effective screening in SOC models (contrary to the usual fractal growth models) and to conjecture that these clusters should be compact also in higher dimensions. In this sense the clusters are more similar to Eden-type clusters rather than to those of the usual fractal growth models (DLA and DBM).

These results are in agreement with the available computer simulations[6] and they suggest that the nature of the critical state in SOC models is very different and much simpler with respect to those of fractal growth models. In fact, according to the present study the SOC models, even if they show a power-law behavior for the distribution of cluster size, they do not seem to give rise to a real anomalous dimension. In this sense they appear similar to an Ising-type model in which the exponent η, that governs the anomalous dimensionality, is intrinsically equal to zero.

We consider here a particular SOC model that is best suited for analytical study.[6,8] This is not a restrictive condition because its properties have been found to be similar to several other models. Given a two-dimensional lattice a quantity E usually named energy can be stored on each site. At a time t (discrete) an input energy δE ($0<\delta E \ll 1$) is added to a randomly chosen site at r, such that

$$E(r,t+1)=E(r,t)+\delta E(r,t)\,. \quad (1)$$

$E(r,t)$ thus assumes non-negative continuous values, and this process repeats itself. One then assumes a limiting threshold $E_{\max}=1$ for the allowed energy on any site. If at a given time t, the energy on the site r is $E(r,t)>1$, a relaxation event occurs and the full amount of energy $E(r,t)$ is distributed, in equal parts, to its $2d$ neighbors. The transferred energy, on its turn, acts as input energy for the neighboring sites. A single input energy event may trigger off relaxation on a set of connected sites; this set is defined as a *cluster*. The boundaries are assumed to be isotropic and, in the thermodynamic limit, at infinity. We adopt free boundary conditions: $E(\text{boundaries},t)=0$. By energy conservation, the transferred energy is eventually let out through the boundaries. We also assume that the process of relaxation is adiabatic in the sense that the system has to relax completely before a new input energy is again introduced. This dynamics leads intrinsically to a critical state with an average energy per site E_c.[6,8] In this critical state the probability distribution that a relaxation event will involve a cluster of size s is a power law $P(s)\simeq s^{-\alpha}$ ($\alpha\simeq 1$ for $d=2$) and therefore clusters of all sizes can be generated.

In order to study the nature of these clusters we have to consider them as the result of a growth process. Once the growth process corresponding to SOC dynamics is identified one can use it within the method of the fixed scale transformation. This method is fundamentally different from the renormalization-group method and it is based on two essential points.[9] The first is the identification of the basic configurations that appear in a fine- (or coarse-) graining process of the structure. In this case the structure is formed by the cluster of sites that have relaxed at least once. These sites are denoted by black dots in Fig. 1. It is actually convenient to consider the intersection of the structure with a line transverse to the growth direction. In this case (as for DLA), for a two-site cell there are just two configurations: type 1 with a black (relaxed) and a white (nonrelaxed) site and type 2 with both sites black. The asymptotic probabilities of occurrence of these configurations (C_1,C_2) can be obtained from the FST considering the dynamical growth process. The fixed-point condition for the FST expresses the invariance of the properties of the structure with respect to displacements of the intersecting line.[9] The FST matrix elements M_{ij} correspond to the conditional probability that a configuration of type i is followed, in the growth direction, by a configuration of type j $(i,j=1,2)$. For example, in Fig. 1 we show the dynamical process for the SOC model corresponding to a starting configuration of type 2 (rectangle). We are going to see that from this probability tree one can derive the matrix elements M_{21} and M_{22}.

Actually the real matrix elements to be used should arise from the convolution of the matrix elements $[M_{ij}(\lambda_n)]$ corresponding to different environments (λ_n), each weighted with the appropriate probability distribution,[9]

$$\tilde{M}_{ij}=\sum_n P(\lambda_n;C_1,C_2)M_{ij}(\lambda_n)\,, \tag{2}$$

where $P(\lambda_n;C_1,C_2)$ gives the probability that the growing zone considered is at a distance λ_n from another branch of the cluster. In general, one has to consider explicitly these fluctuations of the boundary conditions in order to compute the fractal dimension. In the present

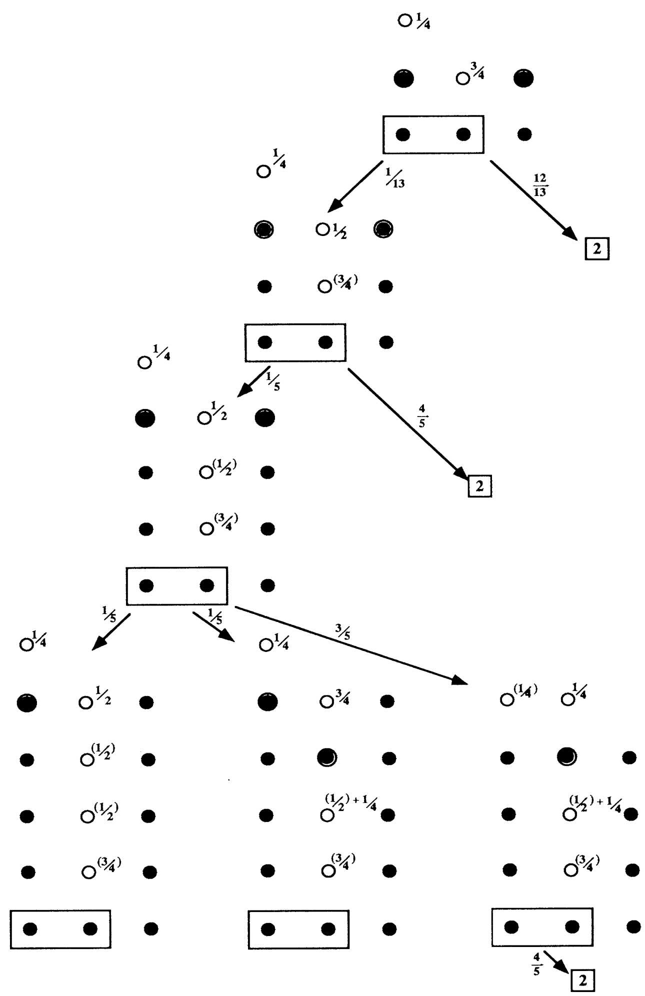

FIG. 1. Scheme of the probability tree for the calculation of the matrix element M_{22} in the case of closed boundary conditions. This matrix element corresponds to the probability that a configuration of type 2 (enclosed in rectangle) with both sites black (relaxed) is followed, in the growth direction, by a configuration of the same type. The circled black sites are those that have just relaxed, while the white sites are the candidates for future relaxation. The numbers near white sites correspond to the energy that has been transferred to these sites by previous relaxations.

case, however, we begin with the simpler question of whether the clusters are compact or fractals. If they would turn out to be compact this would imply self-consistently that $P(\lambda_0)=1$ and $P(\lambda_n)=0$ $(n>0)$. This means that the structure does not leave empty regions because λ_0 is the minimal length considered. In this case it is sufficient to consider only the case of closed boundary conditions (BC) $(\lambda=\lambda_0)$.

By the same reasoning one can also show that if a fully occupied cell (type 2) is certainly followed by another fully occupied cell for $\lambda=\lambda_0$,

$$M_{22}(\lambda_0)=1\ , \tag{3}$$

then this implies self-consistently that $P(\lambda_0)=1$ and $D=d=2$. We will concentrate therefore only on the case of closed BC (omitting the variable λ) and consider the following argument. If M_{22} will converge to the value 1, this (closed BC) is the only boundary condition one has to consider and the treatment is fully self-consistent. If, on the other hand, M_{22} will converge to a value smaller than 1, this will be an indication that the clusters are fractals and our treatment will have a degree of non-self-consistency proportional to the difference $1-M_{22}$. This means that, strictly speaking, we should also include the other types of boundary conditions in the calculation. However, the closer the dimension to the value 2, the less important are these extra types of BC.

For a single type of boundary condition (closed, $\lambda=\lambda_0$) the fixed point for the distribution (C_1,C_2) is simply given by[9]

$$C_1^*=(1+M_{12}/M_{21})^{-1},\quad C_1^*+C_2^*=1\ , \tag{4}$$

and the fractal dimension D is

$$D=1+\ln(C_1^*+2C_2^*)/\ln 2\ . \tag{5}$$

Since $M_{21}=1-M_{22}$ it is clear that if $M_{22}=1$, one has $M_{21}=0$ and one obtains $C_1^*=0$ and $D=2$.

We focus therefore on the study of the matrix element M_{22}. The scheme of this calculation within the FST method is shown in Fig. 1. One starts from a configuration consisting of a pair of black (relaxed) sites. In view of the fact that the boundary conditions are closed ($\lambda=\lambda_0$) this cell is followed on one side by another black site.[9] The growth process corresponding to further relaxations is considered only within the column above the starting cell.

In order to compute asymptotic properties, the growth mechanism should refer to an infinite cluster. As in the percolation problem at p_c the SOC process can generate clusters of all sizes. We therefore have to complement the rules of the SOC propagation of relaxation with a condition of connectivity.[10] In practice we have to make sure that the process we consider would generate, in principle, an infinite cluster. In this respect the first relaxation process is trivial and must occur with equal probability in one of the two sites above the starting cell. Suppose that this is the site on the left; then the relevant side for the boundary condition is the right one as shown in Fig. 1. The site that has just relaxed is encircled and, as a technical simplification, we consider the problem as strictly periodic so that, if an event occurs in the relevant column, a similar event will also occur, at the same time, in the adjacent columns[9] (see Fig. 1). The white sites are the possible candidates for further relaxation and the numbers near them correspond to the energy that they have received from the relaxation processes that have already occurred. The next relaxation process must occur in one of these two sites or in both of them. In view of the connectivity condition we have to exclude the possibility that no relaxation will occur. In order to do this we first compute the probability (non-normalized) for each of these events and then we normalize the final probabilities among all the events that correspond to some relaxation.

The probability W that the white site that receives an energy of $\frac{1}{4}$ will relax can be written as

$$W(\tfrac{1}{4})=\int_{\frac{3}{4}}^{1}P(E)dE\ , \tag{6}$$

where $P(E)dE$ gives the probability that this site had an energy E before the addition of the extra energy $\frac{1}{4}$. The probabilities for the various events P_i $(i=1,2,3,4)$ corresponding to the configuration at the top of Fig. 1 are given by

$$\begin{aligned}&P_1=W(\tfrac{1}{4})W(\tfrac{3}{4}),\quad P_2=[1-W(\tfrac{1}{4})]W(\tfrac{3}{4})\ ,\\&P_3=W(\tfrac{1}{4})[1-W(\tfrac{3}{4})]\ ,\\&P_4=[1-W(\tfrac{1}{4})][1-W(\tfrac{3}{4})]\ .\end{aligned} \tag{7}$$

The meaning of these four possibilities is the following: P_1 corresponds to the case in which both sites relax, P_2 to the relaxation of the site on the right (denoted by $\frac{3}{4}$) and not of the one on the left. P_3 corresponds to the reciprocal situation and P_4 corresponds to no relaxation at all.

In order to satisfy the condition that the growth process should not stop we have to exclude the possibility that P_4 occurs and normalize the other three cases accordingly. This gives the final probabilities $\tilde{P}_i$ $(i=1,2,3)$ that will be used in the calculation of the matrix elements,

$$\tilde{P}_i=\frac{P_i}{\sum_{j=1,2,3}P_j}\ . \tag{8}$$

In order to compute these probabilities explicitly we have to specify the probability density $P(E)$ to be used in Eq. (6). This should be the one corresponding to the critical state. The simplest approximation, that becomes exact for large dimension, is to use a flat distribution $P(E)=1$; E in $[0,1]$. In the present calculations we will use this distribution, but it is possible to show that the use of different distributions, such as those discussed in Ref. 8, leads to essentially the same results. In this respect one may notice that the discussion of the absence of effective screening in SOC models (see later) is basically independent of the distribution used.

For the first nontrivial growth process shown in Fig. 1 we have

$$\tilde{P}_1=\tfrac{3}{13}\ ,\quad \tilde{P}_2=\tfrac{9}{13}\ ,\quad \tilde{P}_3=\tfrac{1}{13}\ . \tag{9}$$

Both $\tilde{P}_1$ and $\tilde{P}_2$ give rise to the occupation of the empty site above the starting cell, leading therefore to a configuration of type 2. In case P_3 occurs one has instead to consider further processes by a similar type of analysis. If a site that has just received a certain amount of energy does not relax it is no longer a candidate for relaxation unless it receives some more energy. In such a case the amount of energy that was not enough to produce relaxation is indicated in parentheses. These features are properly taken into account in the calculation of the corresponding probabilities.

For the matrix element M_{22} we obtain finally

$$\begin{aligned} M_{22}(\mathrm{I}) &= \tfrac{12}{13} = 0.92308\,, \\ M_{22}(\mathrm{II}) &= M_{22}(\mathrm{I}) + \tfrac{1}{13}\cdot\tfrac{4}{5} = 0.98462\,, \\ M_{22}(\mathrm{III}) &= M_{22}(\mathrm{II}) + \tfrac{1}{3}\cdot\tfrac{1}{5}\cdot\tfrac{3}{5}\cdot\tfrac{4}{5} = 0.99200\,, \end{aligned} \qquad (10)$$

where the roman index refers to the order of the calculation and for the third-order term we have only considered the leading contribution (see Fig. 1).

From the results reported in Eqs. (10) there is strong evidence that M_{22} will eventually converge to the value 1. This implies that no holes are left by the SOC process and therefore a compact structure is generated. In fact one obtains $M_{21} = 1 - M_{22} \rightarrow 0$, which inserted into Eq. (4) leads to $C_1^* = 0$, $C_2^* = 1$, and therefore to $D = d = 2$.

We have also computed explicitly the other matrix element M_{12} in order to obtain explicitly the value of D at the various orders of the calculations. This gives

$$D(\mathrm{I}) = 1.9353, \quad D(\mathrm{II}) = 1.9878, \quad D(\mathrm{III}) = 1.9936\,, \qquad (11)$$

and the extrapolation of this value to higher orders strongly points to a convergency towards $D = 2$.

One may wonder about the fact that for a proper analysis of the fractal dimension one should use in the FST the growth rules that are the asymptotic scale-invariant ones corresponding to the given growth rules at the minimal scale.[11] In fact, in order to obtain the fractal dimension D from Eq. (5), it is necessary that the distribution (C_1^*, C_2^*) is the same at all scales, and so should be also the dynamical process that one uses to compute the matrix elements. Here instead we have used directly the growth rules corresponding to the minimal scale. In this case, however, this point is not relevant because the results show that a compact structure is generated already at the minimal scale and this guarantees compactness also at any other scale.

In addition to the convergence evidenced by the explicit calculation, our analysis also allows us to gain insight into the nature of the process. From Fig. 1 one can observe that whenever empty regions are left, these results are very unstable because any perturbation from higher-order processes will induce their relaxation. This situation corresponds to an essential *absence of screening* effects in the spreading of the relaxation process. Given the general nature of this feature one can conjecture that SOC generates compact clusters also in higher dimensions. Therefore, this process is fundamentally different from the usual fractal growth models (DLA, DBM, IP) in which strong screening effects can be identified[9,12] and are crucial in the generation of fractal structures. The SOC clusters appear instead more similar to Eden-type clusters,[9] compact and possibly with an irregular surface.

In conclusion, the present study shows that SOC dynamics, even though it leads to a critical state with a power-law distribution of cluster sizes, generates clusters that are compact objects. Therefore, the SOC critical state does not give rise to a real anomalous dimension and it is, in this sense, of much simpler nature than the critical states corresponding to the asymptotic behavior of the usual fractal growth models like DLA and DBM.

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