Small-Amplitude Periodic and Chaotic Solutions of the Complex Ginzburg-Landau Equation for a Subcritical Bifurcation

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We present numerically obtained bounded solutions of the one-dimensional complex Ginzburg-Landau equation with a destabilizing cubic term and no stabilizing higher-order contributions. The boundedness results from competition between dispersion and nonlinear frequency renormalization. We find chaotic and also stationary and time-periodic states with spatial structure corresponding to a periodic array of pulses. An analytical description is presented. Possibly experimental results connected with the *dispersive chaos* found in binary fluid mixtures can be explained.

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In the last years the complex Ginzburg-Landau equation

$$\partial_t A = [(1+ic_1)\partial_{xx}^2 + 1 - (\alpha+ic_2)|A|^2]A + (\text{higher-order terms}) \quad (1)$$

has become very popular.[1] It describes the small and slowly varying amplitude and phase of a mode that bifurcates via an oscillatory instability from a homogeneous basic state. Equation (1) represents the simplest version for quasi-one-dimensional systems without reflecting boundaries with irrelevant parameters transformed away. This equation is most useful for supercritical bifurcations where one may set $\alpha = 1$. Then it exhibits plane-wave solutions within a stable wave-number range if $1+c_1c_2 > 0$ and turbulent solutions if $1+c_1c_2 < 0$.[1,2] Stable quasiperiodic solutions *(wave trains)* are also found in a rather restricted parameter range near the onset of turbulence.[3] Actually, supercritical Hopf bifurcations do not appear to be easily accessible. In chemical systems, like the Belousov-Zhabotinski reaction, one is apparently never near the bifurcation. The well-known Rayleigh-Bénard instability in binary fluid mixtures is an example, but for the usual liquids the supercritical case occurs in a tiny range of slightly negative separation ratios.[4-6] A promising system is electroconvection in nematics,[7] but a recent experiment has revealed a weakly subcritical bifurcation even there.[8] The most detailed experiments have up to now been performed with the (secondary) oscillatory instability in Rayleigh-Bénard convection of low-Prandtl-number fluids.[3,9]

For subcritical bifurcations, where one may choose $\alpha = -1$, Eq. (1) is usually of less use. Nevertheless, the Ginzburg-Landau equation with complex coefficients was first applied to plane shear-flow instabilities where in fact the bifurcation is strongly subcritical.[10] At first sight one expects blowup of the solutions which can only be avoided by adding at least a fifth-order stabilizing term. The investigation of pulses, fronts, and wave trains for that equation has been an especially active field since the discovery of stable pulses by Thual and Fauve.[11] The strong interest comes mainly from the experimental observation of similar types of states in binary-fluid convection in the subcritical range.[5] There is, however, evidence that bounded turbulent solutions of Eq. (1) should exist with $\alpha = -1$ without higher-order terms, although this has to our knowledge not actually been shown. The evidence comes from work by Bretherton and Spiegel[12] for the case $\alpha = 0$, and from general mathematical arguments put forward especially by Newell.[1] Also, experimental observations of chaotic and pulselike behavior in binary-fluid convection slightly above threshold were

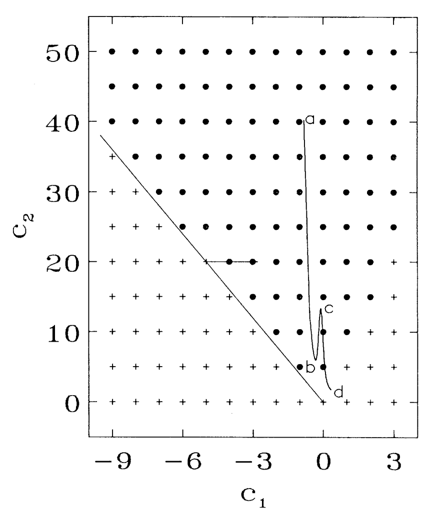

FIG. 1. Classification of the long-time behavior of solutions of Eq. (1) for $\alpha = -1$ in the c_1-c_2 plane. The system length is $L = 2\pi/0.3$ with periodic boundary conditions. The circles correspond to bounded and the crosses to unbounded solutions. The line $c_2 = -4c_1$ separates the two ranges on the left. Along the horizontal line at $c_2 = 20$ we found stable stationary pulses (with wavelength $\lambda = L = 2\pi/2.8$). The curve near $c_1 = 0$ refers to a water-alcohol mixture with varying separation ratio Ψ. At the indicated positions one has a, $\Psi = -7.5\times10^{-5}$; b, -3.5×10^{-4}; c, -0.005; and d, -0.5.

presented recently.[13,14]

We have investigated numerically Eq. (1) with $\alpha=-1$ (without higher-order terms) looking for bounded solutions. Since Eq. (1) is invariant under a simultaneous sign change of c_1 and c_2, we chose $c_2 \geq 0$. The results are shown in Fig. 1. Let us first concentrate on the crosses and circles. The crosses refer to situations where the solutions blow up in finite time (typically after $t \approx 8$), starting from white noise. The circles give the values for c_1 and c_2 where the solution stays bounded at least until $t=60$ (and then for all times—our longest simulation went up to $t=800$). Most of these solutions show spatiotemporal chaos with a qualitative difference for $c_1<0$ and $c_1>0$, analogous to the case $\alpha=0$ (Figs. 3 and 4 in Ref. 12).

However, in some part of the range of bounded solutions the system settles down in arrays of (quasi)stationary and (quasi)periodic pulses. In Fig. 2 we show $R=|A|$ as a function of space and time for $c_1=-4$ and $c_2=20$. In Fig. 2(a) we initially had seven peaks and the solution looks very regular and stays nearly stationary for a long time period except for a very slow drift. Figure 2(b) shows the time evolution for the same parameters where different initial conditions lead to eight peaks, which are not arranged very regularly. They also remain stationary for a long time (until $t \approx 90$) and again between $t \approx 150$ and ≈ 200 after some intermittent behavior. Starting from white noise we observed qualitatively the same behavior, but it takes much longer until a (quasi)stationary situation has evolved. Although we did not get exact periodic and stationary solutions numerically in this fairly long system, these results motivated us to look for an analytic description.

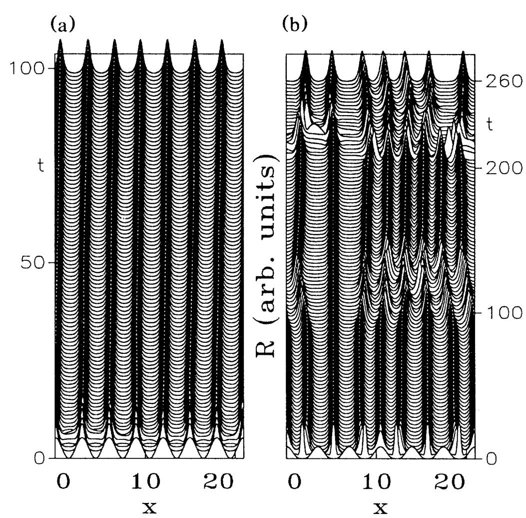

FIG. 2. Space-time plots of the amplitude R for $\alpha=-1$, $c_1=-4$, and $c_2=20$. The system length is $L=2\pi/0.3$ with periodic boundary conditions. We started with the periodic function $A=R_0\cos(kx)$. (a) $R_0=1.6$, $k=2.1$. (b) $R_0=0.8$, $k=1.8$.

For this purpose it seems useful to start from the limit $c_1,c_2 \to \infty$ where Eq. (1) degenerates into the nonlinear Schrödinger equation. This integrable system is well known to have a continuum of soliton solutions which collapses to a discrete set when it is perturbed.[15] One may show this by using solitonic perturbation theory,[16] but we will employ an elementary method. Our treatment is slightly more general then the usual ones since we wish to consider the case of spatially periodic solutions rather than only solitary ones which correspond to the limit of infinitely long period.

Setting $\alpha=-1$ in Eq. (1), and writing

$$A(x,t)=R(x,t)e^{i\Theta(x,t)}, \tag{2}$$

one obtains two real equations which, after suitable linear combination and division through c_1^2, can be written as

$$\varepsilon^2\partial_t R+\varepsilon R\,\partial_t\Theta=[(1+\varepsilon^2)(\partial_{xx}^2-\Theta'^2)+\varepsilon^2+(\beta+\varepsilon^2)R^2]R, \tag{3a}$$

$$-\tfrac{1}{2}\varepsilon\partial_t R^2+\varepsilon^2R^2\partial_t\Theta=(1+\varepsilon^2)\partial_x(R^2\Theta')-\varepsilon[1+(1-\beta)R^2]R^2 \tag{3b}$$

(the primes denote spatial derivatives). Here we have introduced $c_2=-\beta c_1$ and $\varepsilon=1/c_1$. Now an expansion in ε of the form

$$\begin{aligned} R&=R_0+\varepsilon^2R_2+\cdots,\\ \Theta&=\varepsilon^{-1}(\Theta_{-1}+\varepsilon^2\Theta_1+\cdots) \end{aligned} \tag{4}$$

becomes meaningful for sufficiently large values of c_1 [$\beta=O(1)$]. This first leads to the orders ε^{-2} and ε^{-1}, where we have $R_0\Theta'^2_{-1}=0$ and $\partial_x(R_0^2\Theta'_{-1})=0$, respectively. Excluding $R_0=0$, we obtain $\Theta'_{-1}=0$, so that Θ_{-1} only depends on t. Setting $\gamma(t):=\partial_t\Theta_{-1}$ one gets for the next orders

$$0=\partial_{xx}^2R_0-\gamma R_0+\beta R_0^3 \quad (\varepsilon^0), \tag{5a}$$

$$-\partial_x(R_0^2\Theta'_1)=\tfrac{1}{2}\partial_t R_0^2-(1+\gamma)R_0^2-(1-\beta)R_0^4 \quad (\varepsilon^1). \tag{5b}$$

For $\beta,\gamma>0$, Eq. (5a) allows spatially periodic solutions

$$R_0(x,t)=\left[\frac{2\gamma(t)}{[2-m(t)]\beta}\right]^{1/2}\times\mathrm{dn}\left[\left(\frac{\gamma(t)}{2-m(t)}\right)^{1/2}x\,\middle|\,m(t)\right]. \tag{6}$$

Here $\mathrm{dn}(u|m)$ is a Jacobian elliptic function that varies between $(1-m)^{1/2}$ and 1 with period $2K(m)$ [the parameter m is between 0 and 1, and $K(m)$ is the complete elliptic integral of the first kind].[17] For $m \to 1$ the period of dn goes to infinity and (6) degenerates into the pulse $(2\gamma/\beta)^{1/2}\operatorname{sech}(\gamma^{1/2}x)$, while for $m \to 0$ one has small, harmonic oscillations. We expect that the wavelength

$$\lambda = 2K(m(t)) \left[\frac{2-m(t)}{\gamma(t)} \right]^{1/2}$$

of $R_0(x,t)$ is time independent and can be chosen out of a suitable range. This then fixes $m(t)$ once $\gamma(t)$ is known.

To determine γ and some stability properties of (6), we insert R_0 into Eq. (5b) and demand that Θ'_1 also be periodic with λ. Clearly this means that the expression on the right-hand side of Eq. (5b), integrated over one wavelength, is zero. This condition eventually leads to an equation for $\gamma(t)$,

$$D\,\partial_t \gamma = 2E(m)\gamma(1-\gamma/\gamma_0)\,. \tag{7}$$

Here $E(m)$ is the complete elliptic integral of the second kind.[17] Because $E(m) > 0$, the solution with $\gamma = \gamma_0$, i.e., a frequency $\varepsilon^{-1}\gamma_0$ in Eq. (2), is amplitude stable for $D > 0$, and otherwise unstable. We have introduced

$$\gamma_0 = 3\beta \left[(\beta-4) - 2(\beta-1)\frac{1-m}{2-m}\frac{K(m)}{E(m)} \right]^{-1}, \tag{8}$$

which is positive for

$$\beta > \beta_d(m) := 1 + 3\left[1 - 2\frac{1-m}{2-m}\frac{K(m)}{E(m)}\right]^{-1}$$

and diverges at $\beta_d(m)$. Thus $\beta_d = 4$ at $m = 1$ and larger otherwise. At β_d not only the frequency diverges but also the amplitude blows up and, keeping m fixed, the length scale (and thereby the wavelength λ) contracts to zero. Choosing an arbitrary but fixed positive value of λ the solution exists for $\beta > 4$ and degenerates into an array of δ peaks for $\beta \to 4$. In Fig. 1 the line $\beta = 4$ (i.e., $c_2 = -4c_1$), which clearly limits the range of bounded solutions, is included. For D we get

$$D = E(m) + \frac{m}{E(m)/(1-m)K(m) - 2/(2-m)} \left[\frac{m}{(2-m)m}E(m) - \frac{K(m)}{2m} - \frac{1}{2m(1-m)}\frac{E(m)^2}{K(m)} \right], \tag{9}$$

which is positive for all m. Actually, for the solitary limit $m \to 1$ our solutions are included in the formulas presented in earlier work.[18] There, however, attention was focused on the stabilizing higher-order terms in Eq. (1) and on the subthreshold regime $\varepsilon < 0$.

To test the validity of the given expansion, we made numerical simulations of Eq. (1), where λ and c_2 were held fixed varying c_1. The results were compared with the analytical expressions. In Fig. 3 the maximum and the minimum of the stationary solution (6) are plotted as a function of $\beta = -c_2/c_1$ for $\lambda = L = 2\pi/2.8$ (solid lines). The triangles correspond to simulations for $c_2 = 100$ and the circles to $c_2 = 20$. As expected the analytical results are better for larger c_2 (then we also have larger c_1 and smaller ε), but are fairly good over a wide range of c_2. For values of β larger than β_h, with $\beta_h = 7.1$ for $c_2 = 20$ and $\beta_h = 7.8$ for $c_2 = 100$ (termination of symbols in Fig. 3), the pulses became unstable against oscillations. The horizontal line at $c_2 = 20$ in Fig. 1 corresponds to the range of stable, stationary pulses. For larger β the long-time behavior in these short systems either becomes simply periodic or more complex. Within the stationary range there appears to exist a stable wave-number band.

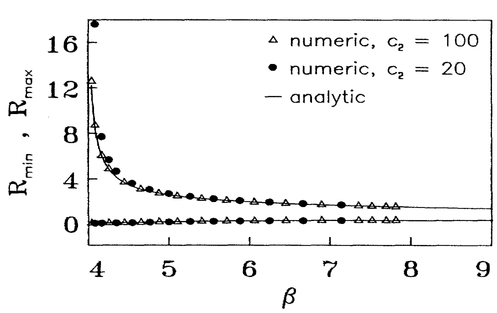

FIG. 3. Minimum R_{min} and maximum R_{max} of stable, stationary pulses as a function of $\beta = -c_2/c_1$ for $L = \lambda = 2\pi/2.8$. The solid line is calculated from Eq. (6). The circles refer to numerical simulations with $c_2 = 20$ and the triangles to $c_2 = 100$.

The curve in Fig. 1 (near $c_1 = 0$) exhibits the c_1-c_2 relation that should be accessible experimentally in water-alcohol mixtures with varying separation ratio Ψ according to calculations with realistic boundary conditions.[6] Possibly the pulses observed in such mixtures slightly above threshold in a rectangular channel[13] with $\Psi = -0.08$ and in an annular channel[14] with $\Psi = -0.069$ are explained by the stationary solutions presented here. Quantitative discrepancies may be due to the higher-order terms left out in Eq. (1). The pulses observed below threshold[5] ($\varepsilon < 0$) are, on the other hand, presumably described better by solutions where fifth-order terms become more important.[11,18] Also the *dispersive chaos* found experimentally for values of Ψ closer to zero[14] is consistent with the behavior we found in the longer systems in the range where the stationary solutions are unstable.

Our investigation has revealed the existence of surprisingly simple (locally) stable stationary solutions of the complex Ginzburg-Landau equation for a subcritical bifurcation. An interesting open problem is the analytical characterization of the upper stability boundary β_h which appears to correspond to a subcritical Hopf bifurcation. We hope that eventually it will also be possible to describe phenomena such as the creation, annihilation, and collision of pulses as observed in experiments in binary fluid mixtures.[14]

It is a pleasure to thank A. C. Newell and P. C. Hohenberg for useful discussions.

[1]See, e.g., A. C. Newell, in *Propagation in Systems Far from Equilibrium,* edited by J. E. Wesfreid, H. R. Brand, P. Manneville, G. Albinet, and N. Boccara (Springer, Berlin, 1988), p. 122, and references therein.

[2]Simulations similar to the ones discussed in this paper showed in a fairly long system ($L=2\pi/0.3$) that chaotic solutions exist even somewhat beyond the line $1+c_1c_2=0$ (on the stable side). All chaotic solutions exhibit strong amplitude variations with phase slips. There is no evidence of pure phase turbulence in the long-time behavior. The scenario is thus analogous to the *defect-mediated turbulence* found in two dimensions [see, e.g., P. Coullet, L. Gil, and J. Lega, Phys. Rev. Lett. **62**, 1619 (1989)].

[3]B. Janiaud, A. Pumir, D. Bensimon, V. Croquette, H. Richter, and L. Kramer (to be published).

[4]See, e.g., J. K. Platten and L. C. Legros, *Convection in Liquids* (Springer, New York, 1983), and references therein.

[5]E. Moses, J. Fineberg, and V. Steinberg, Phys. Rev. A **35**, 2757 (1987); R. Heinrichs, G. Ahlers, and D. S. Cannell, Phys. Rev. A **35**, 2761 (1987); P. Kolodner, D. Bensimon, and C. M. Surko, Phys. Rev. Lett. **60**, 1723 (1988).

[6]W. Schöpf and W. Zimmermann, Europhys. Lett. **8**, 41 (1989).

[7]See, e.g., I. Rehberg, B. L. Winkler, M. de la Torre Juárez, S. Rasenat, and W. Schöpf, Festkörperprobleme **29**, 35 (1989); L. Kramer, E. Bodenschatz, W. Pesch, W. Thom, and W. Zimmermann, Liq. Crys. **5**, 699 (1989).

[8]I. Rehberg, S. Rasenat, M. de la Torre Juárez, W. Schöpf, F. Hörner, G. Ahlers, and H. R. Brand (to be published).

[9]V. Croquette and H. Williams, Phys. Rev. A **39**, 2765 (1989); Physica (Amsterdam) **37D**, 300 (1989).

[10]See, e.g., M. J. Landman, Stud. Appl. Math. **76**, 187 (1987).

[11]O. Thual and S. Fauve, J. Phys. (Paris) **49**, 1829 (1988); W. van Saarlos and P. C. Hohenberg, Phys. Rev. Lett. **64**, 749 (1990).

[12]C. S. Bretherton and E. A. Spiegel, Phys. Lett. **96A**, 152 (1983).

[13]J. J. Niemela, G. Ahlers, and D. S. Cannell, Phys. Rev. Lett. **64**, 1365 (1990).

[14]J. A. Glazier, P. Kolodner, and H. Williams, J. Stat. Phys. (to be published); J. A. Glazier and P. Kolodner (to be published).

[15]A. C. Newell, Rocky Mt. J. Math. **8**, 25 (1978).

[16]For a review see, e.g., Y. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).

[17]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chaps. 16 and 17.

[18]V. Hakim, P. Jakobsen, and Y. Pomeau, Europhys. Lett. **11**, 19 (1990); S. Fauve and O. Thual, Phys. Rev. Lett. **64**, 282 (1990); B. A. Malomed and A. A. Nepomnyashchy, Phys. Rev. A **42**, 6009 (1990).