PHYSICAL REVIEW LETTERS

VOLUME 66 | 6 MAY 1991 | NUMBER 18

Complex Kerr-Newman Geometry and Black-Hole Thermodynamics

J. David Brown, Erik A. Martinez, and James W. York, Jr.
Institute of Field Physics, Department of Physics and Astronomy, The University of North Carolina, Chapel Hill, North Carolina 27599-3255
(Received 5 November 1990; revised manuscript received 26 December 1990)

We establish that in the functional-integral expression for the grand partition function, the thermodynamic properties of a charged, rotating black hole are derived from a complex geometry. The corresponding real "thermodynamical" action is constructed explicitly.

PACS numbers: 04.20.Cv, 05.30.Ch, 97.60.Lf

Recently the proposal[1] to relate the Euclidean action of static black holes to approximations of certain functional integrals that can be interpreted as thermodynamic partition functions has been developed extensively.[2-5] In this paper we extend these developments to the treatment of the stationary geometries of charged, rotating black holes. While there is no real Euclidean metric that represents a rotating black hole, nevertheless, the hole can be described by a complex geometry and its *real* action, which we call the "thermodynamical action."

Periodic imaginary time in Minkowski spacetime was introduced[6] to enable the computation of canonical partition functions by path-integral methods. Subsequent developments apparently led to the belief that the thermodynamical action was always to be obtained from real, positive-definite (Euclidean) metrics. (A complex metric for rotating holes was considered in Ref. 1 but was not employed in later work.[7]) Thus, a real Euclidean metric related to the vacuum rotating hole was obtained by supplementing the analytic continuation $t \rightarrow -it$ of the Boyer-Lindquist stationary time coordinate t by a further parameter transformation $J \rightarrow iJ$, where J is the real angular momentum.[7] However, the resulting metric has little to do with the physical (Lorentzian) Kerr black hole. In this paper we deal with the metric of a charged rotating hole by a different method that leads to a complex metric, and obtain its corresponding *real* thermodynamical action. We address the problem using the canonical formalism for the coupled gravitational and electromagnetic fields. (The case of the uncharged rotating black hole is treated in the Lagrangian formalism in Ref. 8.) Other fields, for example the photons and gravitons resulting from one-loop corrections, could be included in the analysis but are omitted here.

The metric of a constant time slice Σ is denoted by h_{ij}, the associated spatial covariant derivative operator by D_i, the corresponding momentum by P^{ij}, the lapse function by N, and the shift vector by V^i. Thus, the spacetime metric has the form

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt) . \quad (1)$$

The electromagnetic field is described by the one-form $A_\mu dx^\mu$. The canonical field variables are the spatial components A_i of the vector potential and the conjugate momentum $\mathcal{E}^i$, which is $-1/4\pi$ times the electric-field vector density. The component A_0 of the vector potential is the Lagrange multiplier for the Gauss's-law constraint,

$$\mathcal{G} \equiv -D_i \mathcal{E}^i = 0 . \quad (2)$$

The Einstein equations include the Hamiltonian constraint ($G = \hbar = k_B = c = 1$)

$$\mathcal{H} \equiv \frac{16\pi}{\sqrt{h}}[P^{ij}P_{ij} - (P^i_i)^2/2] - \frac{\sqrt{h}}{16\pi}R + \frac{2\pi}{\sqrt{h}}\mathcal{E}^i\mathcal{E}_i + \frac{\sqrt{h}}{4\pi}\partial_{[i}A_{j]}\partial^i A^j = 0 , \quad (3)$$

and the momentum constraint

$$\mathcal{H}_i \equiv -2D_j P_i^j + 2\mathcal{E}^j \partial_{[i} A_{j]} = 0 . \quad (4)$$

The evolution equations of motion are expressed through the Poisson brackets by

$$\partial F/\partial t = \{F, H\} , \quad (5)$$

where F represents any function of the dynamical variables h_{ij}, P^{ij}, A_i, or $\mathcal{E}^i$, and H is the Hamiltonian. The Hamiltonian has the form

$$H = \int d^3x (N\mathcal{H} + V^i \mathcal{H}_i + A_0 \mathcal{G}) + H_{\text{boundary}} , \quad (6)$$

where the boundary terms H_{boundary} are presented in detail in Ref. 9, and are discussed below.

The line element (1) describes a Lorentzian geometry whenever N, V^i, and h_{ij} are real. If N is imaginary, with V^i and h_{ij} real, then the metric (1) is Euclidean. This complexification or "Euclideanization" can be made explicit by replacing N with $-iN$ in the metric and equations of motion, where N is then taken to be real. However, this is *not* the complexification that is appropriate for gravitational thermodynamics: The correct complexification also includes changing the shift vector V^i and the potential A_0 from real to imaginary. We thus define the following complexification map Φ of the field variables:

$$\Phi(N) = -iN , \quad (7)$$

$$\Phi(V^i) = -iV^i , \quad (8)$$

$$\Phi(A_0) = -iA_0 , \quad (9)$$

$$\Phi(F) = F , \quad (10)$$

where N, V^i, A_0, and the dynamical variables F are real. Observe that *the Cauchy data (F) are invariant* with respect to the complexification Φ, while *the Lagrange multipliers* N, V^i, and A_0 *receive a complex phase.* We regard this as a key principle that will apply also to other fields.[10]

The relevance of the complexification Φ comes from the observation that the Hamiltonian (6), including the boundary terms presented below, is a sum of terms that are linear in the Lagrange multipliers; it is therefore mapped to

$$\Phi(H) = -iH . \quad (11)$$

For stationary histories of the system, the left-hand side of the equations of motion (5) is zero. Therefore Φ *preserves the constraints (2)-(4) and the dynamical equations of motion (5) of stationary spacetimes.*

Now suppose the gravitational and electromagnetic fields are those of the Lorentzian Kerr-Newman solution, described with respect to stationary Boyer-Lindquist time slices Σ. Then Φ maps this solution to a complex history that satisfies the field equations and defines a saddle point in an appropriate functional integral. Under the action of Φ, the metric (1) becomes the complex "Kerr-Newman metric"

$$\Phi(ds^2) = N^2 dt^2 + h_{ij}(dx^i - iV^i dt)(dx^j - iV^j dt) , \quad (12)$$

while the electromagnetic field is described by

$$\Phi(A_\mu dx^\mu) = -iA_0 dt + A_i dx^i . \quad (13)$$

Because the energy, angular momentum, and electric charge are defined by two-surface integrals of the Cauchy data, they remain real with their physical values. Furthermore, (12) has an ergosurface outside of the horizon (or "bolt"[7]) just as does the real Lorentzian black-hole geometry. This feature cannot be incorporated into a real positive-definite metric.

Consider the rotating axisymmetric black hole and its surroundings contained in a cavity with axisymmetric boundary two-surface B. We regard the interior of B as the thermodynamic system and the exterior as the heat bath. The heat bath that equilibrates the system has a constant angular velocity[11] ω^* as measured with respect to the "fixed stars" at spatial infinity. The zero-angular-momentum observers (ZAMO's),[12] who are at rest in the slices of constant stationary time, have an angular velocity ω^*_{ZAMO} with respect to the fixed stars. Consequently, the angular velocity of the heat bath with respect to the ZAMO's, using ZAMO proper time, is

$$\hat{\omega} = N^{-1}\omega = N^{-1}(\omega^* - \omega^*_{\text{ZAMO}}) . \quad (14)$$

ω^*_{ZAMO} and hence $\hat{\omega}$ depend on position.[12]

The shift vector, in its role as a Lagrange multiplier for the gravitational momentum constraint, embodies the ZAMO-measured "chemical potential" ω associated with the angular momentum of the system. It is given by

$$V^i = \omega\phi^i = (\omega^* - \omega^*_{\text{ZAMO}})\phi^i , \quad (15)$$

and $\hat{V}^i = N^{-1}V^i = \hat{\omega}\phi^i$ is the proper ZAMO-measured spatial velocity of the heat bath. With V^i given by (15), regularity of the complexified geometry (12) now fixes the constant angular velocity of the heat bath to be that of the horizon, ω^*_H. (This complex geometry is equivalent to that of Ref. 1.) One can show that the world lines of the elements of the bath follow the orbits of a stationary Killing vector[13] that is, in fact, the "corotating" Killing vector.[14]

The ZAMO's have four-velocities u^μ which are the unit normals of the time slices. In hydrodynamic terms, they are "Eulerian" observers at rest in the time slices, and watch the heat bath rotate past. Correspondingly, the "Lagrangian" observers with four-velocities w^μ are comoving with the heat bath. These observers are useful in describing the heat bath locally, but not globally, because bath rotation implies $w_{[\mu}\partial_\nu w_{\alpha]} \neq 0$ and there is no global time slicing with respect to which the Lagrangian observers are at rest. We therefore must adopt the Eulerian viewpoint for a global description of dynamical and thermodynamical states and use ZAMO-measured variables.[15]

The inverse Hawking temperature at infinity for a stationary black hole is given by $\beta_\infty = 2\pi\kappa^{-1}$, where

$$\kappa = [(D^i N)(D_i N)]^{1/2}|_H \tag{16}$$

is the constant surface gravity of the horizon. Thus, the proper inverse temperature determined by a ZAMO is

$$\beta = N\beta_\infty \,. \tag{17}$$

[The proper inverse temperature determined by a Lagrangian observer at the same event is β/γ, where $\gamma = -g(u,w)$.]

We will construct the thermodynamical action for a grand canonical ensemble, in which the inverse temperature and chemical potentials are specified on the topologically two-spherical boundary B in the stationary time slice Σ. This means that the two-geometry, the tangential components of A_i, and the Lagrange multipliers N, V^i, and A_0 are all fixed on B. The boundary terms in the Hamiltonian that reflect these conditions are

$$H_B = \oint_B d^2x\sqrt{\sigma}\left[\frac{2}{\sqrt{h}} n_i V_j P^{ij} + \frac{1}{8\pi} N(k - k^0) + \frac{1}{\sqrt{h}} A_0 n_i \mathcal{E}^i\right], \tag{18}$$

where σ is the determinant of the metric induced on B, and n^i is a "radially" outward-pointing unit normal defined on Σ. Also, k is the trace of the extrinsic curvature of B as embedded in Σ, and k^0 is a constant chosen to ensure that the action for flat spacetime is zero. These boundary terms can be derived[9] by observing that, from a Lagrangian point of view,[8] the grand canonical boundary conditions correspond to fixing the three-geometry of ${}^3B$ and the projection of A_μ onto ${}^3B$, where ${}^3B$ is the history of the boundary B. (The intersection of ${}^3B$ with Σ is the two-surface B.) The total action in that case includes a boundary term that is proportional to the integral over ${}^3B$ of the trace of the extrinsic curvature of ${}^3B$. Introducing a foliation by spacelike slices and performing a 3+1 split, one finds that this action has the standard canonical form with a Hamiltonian that includes the surface terms (18).

The Hamiltonian (6) also includes boundary integrals over H, which is the intersection of the outer event horizon with Σ for the Lorentzian black hole (1) and the bolt for the complexified geometry (12). The metric induced on H is not to be fixed, but rather its "momentum" associated with a radial foliation.[4] The corresponding boundary terms are equal to [9]

$$H_H = -\oint_H d^2x\sqrt{\sigma}\left[\frac{2}{\sqrt{h}} n_i V_j P^{ij} + \frac{1}{8\pi} n^i D_i N + \frac{1}{\sqrt{h}} A_0 n_i \mathcal{E}^i\right]. \tag{19}$$

The full Hamiltonian is given by Eqs. (6), (18), and (19) where $H_{\text{boundary}} = H_B + H_H$.

The action has canonical form with Hamiltonian (6). For stationary histories, the "$p\dot{q}$" terms vanish, and upon application of Φ the action becomes purely imaginary. The thermodynamical action I^* is obtained from the identification of phases $\exp(-I^*) \equiv \exp(iI)$ in the functional integral and is real. I^* includes a periodic identification of the Boyer-Lindquist time slices Σ, with period β_∞. This action is further simplified by eliminating the constraints (2)-(4). This procedure[3] places the action in the form of a variational *principle,* whose extrema relate the given boundary data to those constants or functions that remain as degrees of freedom to be varied after the constraints have been eliminated.[5] The resulting action I^* is given by β_∞ times the sum of the right-hand sides of Eqs. (18) and (19).

Now recall the definitions (14) and (17) of the proper local inverse temperature and angular velocity, and further define the proper electrostatic potential determined by a ZAMO to be

$$\hat{\varphi} \equiv -A_\mu u^\mu = -N^{-1}A_0 + \hat{\omega} A_i \phi^i \,. \tag{20}$$

The action then has the explicit form

$$I^* = \oint_B d^2x\sqrt{\sigma}\beta(k - k^0)/8\pi - \oint_H d^2x\sqrt{\sigma}\beta n^i D_i N/8\pi N - \oint d^2x\sqrt{\sigma}\beta[\hat{\omega} 2 n_i \phi_j P^{ij}/\sqrt{h} + \hat{\varphi} n_i \mathcal{E}^i/\sqrt{h} + \hat{\omega}(A_i\phi^i) n_i \mathcal{E}^i/\sqrt{h}]|_H^B \,. \tag{21}$$

In the first of the surface integrals at H, the normal n_i is proportional to the gradient $D_i N$ of the lapse function. Then using expression (16) for the surface gravity shows that this term is $-A_H/4$, minus the black-hole entropy, where A_H is the area of the event horizon. The remaining integrals at H all vanish: Those involving $\beta\hat{\omega}$ are zero because $\omega^*_{\text{ZAMO}} \to \omega^*_H$ as $B \to H$;[14] those involving $\beta\hat{\varphi}$ are then zero as required by regularity of the electromagnetic field for the complexified Kerr-Newman geometry.

The thermodynamical content of the boundary integrals at B can be recognized as follows. The quasilocal mass-energy of a stationary gravitating system is given by the two-surface integral (Ref. 5, footnote 14)

$$E = \frac{1}{8\pi}\oint_B d^2x\sqrt{\sigma}(k - k^0)\,, \tag{22}$$

while the total angular momentum is defined by an Arnowitt-Deser-Misner surface integral[16]

$$J = -\oint_B d^2x\sqrt{\sigma} 2 n_i \phi_j P^{ij}/\sqrt{h}\,, \tag{23}$$

and the total electric charge is given by Gauss's law

$$Q = -\oint_B d^2x\sqrt{\sigma} n_i \mathcal{E}^i/\sqrt{h}\,. \tag{24}$$

From these expressions for E, J, and Q one obtains the proper differential surface fluxes dE/dA, dJ/dA, and dQ/dA. The thermodynamical action is now

$$I^* = -\tfrac{1}{4}A_H + \oint_B d^2x\sqrt{\sigma}\left[\beta\frac{dE}{dA} - (\beta\hat{\omega})\left(\frac{dJ}{dA} + A_\phi\frac{dQ}{dA}\right) + (\beta\hat{\varphi})\frac{dQ}{dA}\right]. \tag{25}$$

Consider a choice for the two-surface B. In general, it is not possible that the grand canonical boundary data β, $\beta\hat{\omega}$, and $\beta\hat{\varphi}$ are all constant on B. This can be seen from an examination of the Kerr-Newman metric. We could choose B so that one of these thermodynamical variables is constant, and then the corresponding term in I^* is a product of constant surface data. For example, if B is an isothermal surface β=const, then the energy term in I^* becomes $\beta E|_B$. If B is an equipotential surface $\beta\hat{\varphi}$ =const, then the corresponding term is $(\beta\hat{\varphi})|_B Q$. If B is a constant-$\beta\hat{\omega}$ surface, then the corresponding terms in I^* are $-(\beta\hat{\omega})|_B\bar{J}$, where $\bar{J}\equiv\oint d^2x\sqrt{\sigma}(dJ/dA + A_\phi dQ/dA)$. [Note that Q and $\bar{J}$ are independent of B by virtue of the Gauss's-law constraint (2) and the momentum constraint (4).] The key point is that for any choice of the boundary surface B, only *surface data* remain in I^*; this is a consequence of formulating thermodynamics in the context of a generally covariant theory.[4]

The term in expression (25) for the thermodynamical action involving the canonical angular momentum J gives the "thermodynamic Massieu potential"[4] associated with the injection of matter into the system. For a sufficiently large boundary surface B, $\omega^*_{\rm ZAMO}$ is negligible and $\omega \approx \omega^*_H$. The term involving $\hat{\varphi}$ gives the potential for the injection of charge. The expression involving the magnetic three-vector potential A_ϕ gives the potential associated with changing the electromotive force around the hole.[17] A detailed discussion of the various thermodynamic potentials will be given elsewhere.

We conclude that all of standard black-hole thermodynamics can be obtained as a stationary-phase approximation to functional-integral representations of partition functions.[3-5]

The authors thank W. Israel and J. Louko for helpful suggestions that led to revision of the original version of this paper. They also thank H. W. Braden and B. F. Whiting for discussions in the early phases of this work, and G. Hayward, D. N. Page, and W. H. Zurek for stimulating discussions. Research support was received from National Science Foundation Grants No. PHY-8407492 and No. PHY-8908741.

[1]G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).

[2]J. W. York, Phys. Rev. D **33**, 2092 (1986); H. W. Braden, B. F. Whiting, and J. W. York, Phys. Rev. D **36**, 3614 (1987).

[3]B. F. Whiting and J. W. York, Phys. Rev. Lett. **61**, 1336 (1988).

[4]J. D. Brown, G. L. Comer, E. A. Martinez, J. Melmed, B. F. Whiting, and J. W. York, Class. Quantum Grav. **7**, 1433 (1990).

[5]H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. York, Phys. Rev. D **42**, 3376 (1990).

[6]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[7]G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. **66**, 291 (1979); S. W. Hawking, in *General Relativity*, edited by S. W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979).

[8]J. D. Brown, E. A. Martinez, and J. W. York, in Proceedings of the Sixth Florida Workshop on Nonlinear Astronomy, edited by S. Detweiler and J. Ipser (New York Academy of Sciences, New York, to be published).

[9]J. D. Brown and J. W. York, "Hamiltonian Dynamics of Spacetime and Gauge Fields in a Bounded Region" (to be published).

[10]In the complex rotation $\exp(i\alpha)$, $\alpha = -\pi/2$, given by (7), Φ acts on the square root $N = (-g^{00})^{-1/2}$ of the spacetime metric $g_{\mu\nu}$. Hence, Φ can also be regarded as acting directly on a tetrad representation of the metric. Because fermions couple to tetrads, our viewpoint underlies the formulation of the flat-spacetime "Euclidean" Dirac fermion by M. R. Mehta, Phys. Rev. Lett. **65**, 1983 (1990). Mehta describes his method as a direct continuation of the *metric,* but his use of double-angle functions in the flat spacetime g_{00} reveals the relation to (7).

[11]See T. Zannias and W. Israel, Phys. Lett. **86A**, 82 (1981). We accept the argument of these authors embraced by their equations (4)–(6), including the first un-numbered equation below (6). This argument implies ω^*=const for *unconstrained* thermal equilibrium. The remainder of their discussion is not relevant to our work because it involves a constraint on momentum exchanges in the heat bath beyond simple conservation of energy and momentum. W. Israel (private communication) called our attention to this reference and to the inappropriateness of constrained equilibrium in this problem.

[12]J. M. Bardeen, in *Black Holes,* edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973).

[13]W. Israel and J. M. Stewart, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), Vol. II.

[14]B. Carter, in *General Relativity* (Ref. 7).

[15]L. Smarr and J. W. York, Phys. Rev. D **17**, 2529 (1978); see Sec. IIB and footnote 13.

[16]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).

[17]K. S. Thorne, R. H. Price, D. A. Macdonald, W. M. Suen, and X. H. Zhang, in *Black Holes: The Membrane Paradigm,* edited by K. S. Thorne, R. H. Price, and D. A. Macdonald (Yale Univ. Press, New Haven, 1986), p. 108.