Universal Limit of Spiral Wave Propagation in Excitable Media

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(Received 10 December 1990)

We demonstrate that for a large class of excitable media the minimum excitability required for spiral wave propagation follows the universal scaling law $\Delta_{\min} = \text{const} \times \varepsilon^{1/3}$, where ε is the usual small parameter associated with the abruptness of excitation. The prefactor of the scaling law is obtained by solving the free-boundary problem of wave propagation. The quantitative validity of the free-boundary formulation is tested for the first time by direct comparison with the results of numerical simulation of a specific two-variable model of excitable kinetics.

PACS numbers: 82.40.Fp, 87.90.+y

Rotating spiral waves are robust patterns found in a wide range of quasi-two-dimensional excitable media. They can be seen as colored waves of oxidation propagating in the Belousov-Zhabotinski (BZ) reaction[1,2] and waves of cyclic-AMP signaling propagating in social amoeba colonies of *Dioctyostelium discoideum.*[3] More indirectly, they have been measured as circulation waves of neuromuscular activity propagating in the heart muscle,[4] and theories have been proposed to explain heart pathologies in terms of these waves.[2,5]

Theoretical investigations of wave propagation in excitable media are commonly based on two-variable reaction-diffusion models of the form[6]

$$\varepsilon \frac{\partial u}{\partial t} = \varepsilon^2 \nabla^2 u + f(u,v)\,, \tag{1a}$$

$$\frac{\partial v}{\partial t} = r\varepsilon \nabla^2 v + g(u,v)\,, \tag{1b}$$

where the functions $f(u,v)$ and $g(u,v)$ describe the nonlinear kinetics of specific systems. In the case of chemical reactions $r = D_u/D_v$ is the ratio of diffusion coefficients of the two components, while for neuromuscular tissues $r=0$. Most generally, the u nullcline $[f(u,v)=0]$ is an N-shaped function and the v nullcline $[g(u,v)=0]$ is monotone, intersecting the u nullcline at only one point (u_0,v_0) as illustrated in Fig. 1. This fixed point (u_0,v_0) is linearly stable but excitable in the sense that a finite-amplitude perturbation can cause the system to execute a large excursion in phase space before returning to equilibrium. When the "excitability" of the medium is altered, this change will have the effect of altering the location of the rest state (u_0,v_0). It is convenient to measure the excitability of the system by the parameter Δ, defined here in terms of Fig. 1 as $\Delta = v^* - v_0$, where v^* corresponds to the value on the v axis at which the propagation velocity c_0 of the planar pulse structure vanishes [i.e., $c(v^*)=0$].

Spiral wave rotation takes place at constant frequency ω, the spiral tip following a circular trajectory of radius R around an "effective" hole region.[6] A central problem of wave dynamics in excitable media is to understand what are the regions of parameter space—of the (ε,Δ) plane—where steadily rotating spiral waves can exist. An important mechanism which limits their existence is linked to the observation that R can become extremely large over some range of the parameters ε and Δ. When R becomes too large, waves necessarily encounter the boundaries of the system where they become absorbed and disappear. Recently, Pelce and Sun[7] studied the large-radius limit of a piecewise linear model of excitable kinetics with $r=0$. By numerically solving the free-boundary formulation of this model for values of ε ranging approximately between 0.1 and 7, they found that R diverges as the excitability of the medium approaches a minimum value $\Delta_{\min}(\varepsilon)$ which is a monotonously increasing function of ε. Furthermore, these authors derived by analytical means a condition fixing the tangential velocity of spiral waves (i.e., ωR) in the large-R limit.

In this Letter, we investigate the small-ε analytical behavior of $\Delta_{\min}(\varepsilon)$ and show that, for the general class of models of the form (1) with $r=0$, it is given by

$$\Delta_{\min} = (g^*/B\alpha^2)^{1/3}\varepsilon^{1/3}\,, \tag{2}$$

where the constants α and g^* are related to the form of the functions f and g, and $B=0.535\ldots$ is a numerical constant. This result is obtained using the free-boundary formulation of wave propagation and is valid in the limit $\varepsilon \ll 1$ which is precisely the one of practical relevance.[6] It has two important consequences. First, it shows that the scaling behavior of $\Delta_{\min}(\varepsilon)$ is universal to the extent

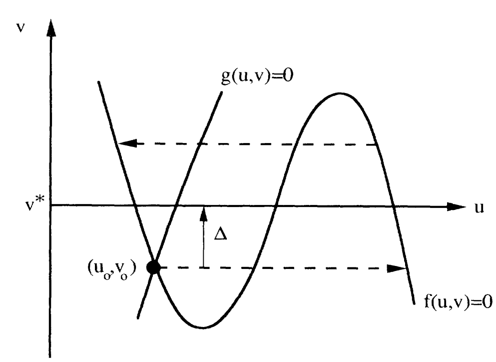

FIG. 1. Typical phase plane for an excitable medium.

that the $\varepsilon^{1/3}$ power law does not depend on details of the excitable kinetics, as long as f and g have the general form characteristic of excitable media. Second, it implies that spiral waves in the neighborhood of the line $\Delta_{\min}(\varepsilon)$ have a speed and a tip structure scaling, respectively, as $\varepsilon^{1/3}$ and $\varepsilon^{2/3}$. This scaling is analogous to the one originally proposed by Fife,[8] but has a different origin here. Fife considered the diffusive case $r=1$ and proposed that the added effect of diffusion would tend to smooth out the discontinuities in v which arise when taking the limit $\varepsilon \to 0$ keeping Δ of $O(1)$.[8] Here, it is the fact that Δ must scale as $\varepsilon^{1/3}$ for very-large-radius spiral waves which is responsible for this scaling. An important consequence of this scaling is that the characteristic size of the spiral tip structure $[O(\varepsilon^{2/3})]$ is large compared to the front thickness $[O(\varepsilon)]$. The complete free-boundary formulation of wave propagation is therefore expected to be most quantitatively accurate in this limit.[6-8] The first direct quantitative test of the formulation is presented in this Letter.

The origin of the free-boundary formulation can best be understood by considering the shape of a planar pulse propagating in the unperturbed medium at constant velocity $c_0 \equiv c(v_0)$. The pulse consists of a narrow front region of width ξ of $O(\varepsilon)$ which results from the abrupt rise of excitation, an excited region of width W, a narrow back region of width ξ, and a refractory region during which the system slowly returns to equilibrium. During the duration of excitation, the system evolves along the rightmost branch of the u nullcline spending a total time T_e of $O(\Delta)$ on this branch. The width $W=c_0T_e$ is therefore of $O(c_0\Delta) \gg \xi$ in the limit $\varepsilon \ll 1$. The front and back regions can therefore be regarded as sharp boundaries separating the enclosed domain of excitation Ω from the equilibrium and recovery regions. The problem of wave propagation becomes a free-boundary problem where the normal velocity of the boundary c_n depends on v and the local interfacial curvature κ via the relation[6] $c_n = c(v) - \varepsilon\kappa$. The field v evolves according to Eq. (1b) with u replaced by $h_e(v)$ inside Ω and $h_r(v)$ outside; $u=h_e(v)$ and $u=h_r(v)$ correspond, respectively, to the rightmost and the leftmost branch of the u nullcline.

As shown in Ref. 7, large-radius spiral waves which exist for values of Δ slightly larger than $\Delta_{\min}$ have a far tip structure (i.e., the structure asymptotically far from the tip region) with a shape corresponding to the involute of a circle.[9] These waves have an *a priori* unknown tangential tip velocity ωR slightly smaller than c_0 which needs to be determined by matching two solutions describing, respectively, the far tip region and the tip region. At lowest order in $1/R$ this matching procedure yields a tangential velocity equal to c_0.[7] A higher-order matching analysis would in principle give the slight reduction of ωR from its lowest-order value c_0 at large R. As we shall show below, the "marginal wave" which corresponds to Δ exactly equal to $\Delta_{\min}$ has a far tip structure in uniform translation at velocity c_0. This wave is therefore entirely describable by the equation for the tip region and no matching procedure is necessary. It can be pictured physically as a wave front with one broken end of the front as depicted in Fig. 2. In addition, the field v in the recovery region does not affect the motion of the front boundary. Consequently, only spatial variations of v within the enclosed domain Ω need to be considered. The free-boundary problem for the marginal wave translating at constant velocity c_0 can be written as

$$c_0 - \varepsilon\kappa^+ = c_0 \sin(\theta^+) , \tag{3a}$$

$$-c(v^-) - \varepsilon\kappa^- = c_0 \sin(\theta^-) , \tag{3b}$$

$$c_0 \partial_z v + g(h_e(v), v) = 0 , \tag{3c}$$

where v obeys (3c) in Ω, (3a) and (3b) express the normal velocity relation of the shape-preserving front $[z^+(x)]$ and back $[z^-(x)]$ boundaries, respectively, and v^- is the value of v evaluated on the back boundary. Here, θ^+ (θ^-) measures the angle between the x axis and the outward (inward) pointing normal on the front (back) boundary, and $\kappa^{\pm} = \pm d\theta^{\pm}/ds^{\pm}$ is the curvature measured along the arclength coordinate $s^{\pm}$ of each boundary.

The problem we are seeking to solve consists in determining the value $\Delta = \Delta_{\min}$ for which a physically admissible solution of the free-boundary problem (3) exists. Consider the solution of (3). The front boundary $z^+(x)$ can easily be obtained by integrating (3a) analytically. It is given implicitly by

$$(c_0/\varepsilon) z^+(\theta^+) = -\ln[1-\sin(\theta^+)] , \tag{4a}$$

$$(c_0/\varepsilon) x(\theta^+) = \tan(\pi/4 + \theta^+/2) - \theta^+ - 1 , \tag{4b}$$

where θ^+ varies between 0 and $\pi/2$. Equations (4) imply that $z^+(x) \to (\varepsilon/c_0)\ln(c_0 x/\varepsilon)$ as $x \to \infty$. The far tip region differs in shape from the planar pulse structure but has the same translation velocity. Thus, at large x the width of the excited region $z^+(x) - z^-(x) \to W$,

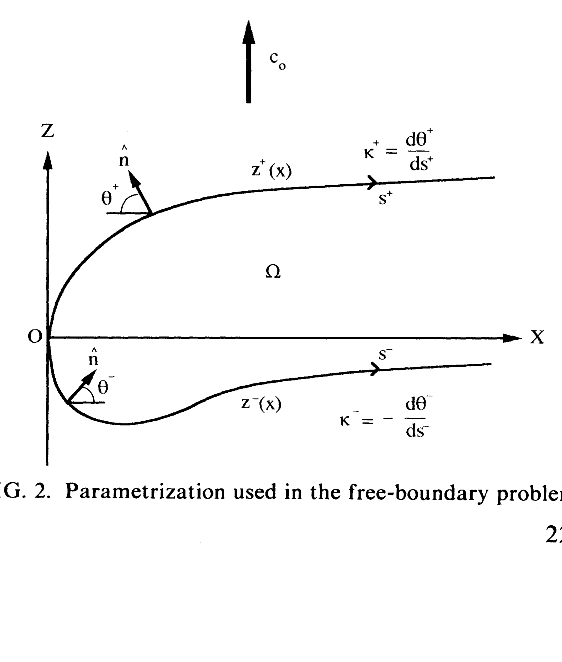

FIG. 2. Parametrization used in the free-boundary problem.

where W is the width of the excited region of the planar pulse structure. At the wave tip $[z^{\pm}(0)=0,\ \theta^{\pm}=0]$, Eqs. (3a) and (3b) imply that $v^{-}(0)=v_0$ and $\kappa_{\rm tip}=\kappa(0)=c_0/\varepsilon$. For a physically smooth solution to exist we expect that $1/\kappa_{\rm tip}$ and W should be of the same order. Since W scales as $c_0\Delta$ and $1/\kappa_{\rm tip}$ as ε/c_0, the ratio $c_0^2\Delta/\varepsilon$ should be of $O(1)$. Furthermore, close to v^*, $c(v)=\alpha(v^*-v)+O((v^*-v)^2)$, where $\alpha\equiv(dc/dv)_{v=v^*}$ is a constant uniquely determined by $f(u,v^*)$. The requirement that $c_0^2\Delta/\varepsilon=\alpha^2\Delta^3/\varepsilon$ be of order unity gives the scaling $\Delta\sim\varepsilon^{1/3}$ and $1/\kappa_{\rm tip}\sim\varepsilon^{2/3}$.

We now show that (3) can be written in a form where only the ratio $B=g^*\varepsilon/\alpha^2\Delta^3$ appears explicitly, and determine its value. The solution of (3a) is already given by (4). The solution of (3c) inside Ω is given implicitly by $z=-c_0\int dv/g(h_e(v),v)+\text{const}$. Since Δ scales as $\varepsilon^{1/3}\ll 1$, the function $g(h_e(v),v)$ in the denominator of the integrand can be replaced by a constant value: $g(h_e(v),v)=g^*+O(\varepsilon^{1/3})$, where $g^*\equiv g(h_e(v^*),v^*)$. Performing the integral for z over the appropriate limits we obtain at once $v=v_0-(g^*/c_0)[z-z^+(x)]$ in Ω, and $v^-=v_0-(g^*/c_0)[z^-(x)-z^+(x)]$ on the back boundary. Substituting the expression for v^- in the velocity relation $c(v^-)=\alpha(v^*-v^-)$ and using the definitions $\Delta=v^*-v_0$ and $c_0=\alpha\Delta$, the equation for the back boundary can be rewritten in the form

$$-\alpha\Delta\{1+(\alpha\Delta^2/g^*)^{-1}[z^-(x)-z^+(x)]\}-\varepsilon\kappa^- =\alpha\Delta\sin(\theta^-)\,.$$

Finally, performing the scale change $(X,Z^{\pm})=(\alpha\Delta/\varepsilon x,\ \alpha\Delta/\varepsilon z^{\pm})$ and $K^{\pm}=(\alpha\Delta/\varepsilon)^{-1}\kappa^{\pm}$, we obtain the final form of (3b):

$$-1-B\{Z^-(x)-Z^+(x)\}-K^-=\sin(\theta^-)\,, \qquad (5)$$

where $Z^+(X)$ is defined implicitly by (4). A physically admissible solution of (5), with $Z^-(0)=0$ and $Z^+(X)-Z^-(X)\to\text{const}=2/B$ for $X\to\infty$, was found numerically to exist only for a unique value of B ($B=0.535\ldots$). The constraint of a unique B fixes the value of Δ ($\Delta=\Delta_{\min}$) for the marginal wave and yields at once the scaling form (2).

Since the characteristic size of the tip structure scales as $\varepsilon^{2/3}$ which is much larger than the front thickness which scales as ε, the resulting prediction (2) for $\Delta_{\min}$ is expected to be quantitatively accurate in the limit of small ε. To test this prediction, and thus the validity of the free-boundary formulation, we have compared the value of $\Delta_{\min}$ given by (2) with the value obtained by direct numerical simulation of a simple Fitz-Hugh-Nagumo form of excitable kinetics. This kinetics is given by $f(u,v)=3u-v-u^3$ and $g(u,v)=u-\delta$. With this choice we have $v^*=0$, $(u_0,v_0)=(\delta,3\delta-\delta^3)$, $\Delta=\delta^3-3\delta$, $g^*=2\sqrt{3}$, and $\alpha=1/\sqrt{2}$. Combining the above values we obtain the prediction for this choice of kinetics: $\Delta_{\min}=2.35\varepsilon^{1/3}$. The system of partial differential equations (1) with $r=0$ and the above choice of f and g was integrated numerically using an ADI (alternating direction implicit) finite-difference algorithm. Zero-flux boundary conditions were used at the edges. We performed the test for $\varepsilon=10^{-3}$ which required using large 512×512 square lattices. With $dx=0.333\varepsilon$ and Δ near $\Delta_{\min}$, the length of the simulation box $L=512\ dx$ is about 8 times the width of the excited region of the planar pulse. A planar pulse structure smoothed out at one end was used as the initial condition. After an initial transient lasting about 10^3 times steps ($dt=0.2\varepsilon$) the wave tip (defined by $u=\partial_t u=0$) rotated uniformly on a circular track of radius R_0. Here, R_0 should be interpreted as the "zero-cycle radius" of the tip trajectories in the sense that, for large radii, waves disappear at the edges before a complete rotation can be completed. Because of the slow return to equilibrium of the recovery variable, R_0 will differ from the "many-cycle radius" R for the smaller values of R_0 measured here. However, the value of $\Delta_{\min}$ at which R_0 is infinite (i.e., the quantity we wish to determine) is obviously identical to the value at which R is infinite. In good agreement with theoretical expectations, the predicted value $\Delta_{\min}=0.235$ (for $\varepsilon=10^{-3}$) and the "measured value" $\Delta_{\min}=0.248$ at which $1/R_0=0$ (Fig. 3) only differ by an amount which is comparable to the higher-order corrections $[O(\varepsilon^{2/3})]$ to the predicted value. For $\Delta=0.248$ we obtained a translating wave with a boundary shape ($u=0$ contour) in good agreement with the shape determined by (4) and (5).

Finally, for $\Delta<0.248$ we observed "retracting waves" characterized by a forward propagation at constant velocity, and a retraction of the tip structure (in the $+x$ direction in terms of Fig. 2) causing the lateral width of the wave to shorten in time. When the width of the excited region becomes comparable to the front thickness, which occurs when Δ is of order $\varepsilon^{1/2}\ll\varepsilon^{1/3}$ ($\Delta\ll\Delta_{\min}$), the planar pulse structure becomes unstable. Since the far tip region of retracting waves is identical to the planar pulse structure, these waves themselves become unstable in this limit.

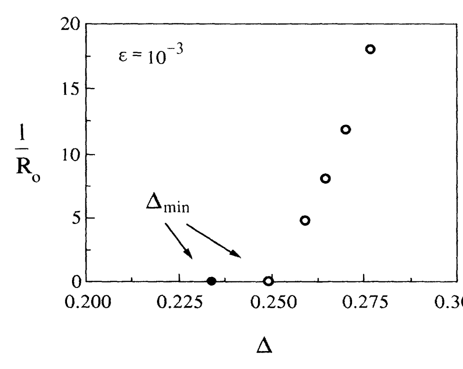

FIG. 3. Open circles represent the values of $1/R_0$ obtained by simulation of the Fitz-Hugh-Nagumo model. The solid circle represents the analytically predicted value $\Delta_{\min}=0.235$.

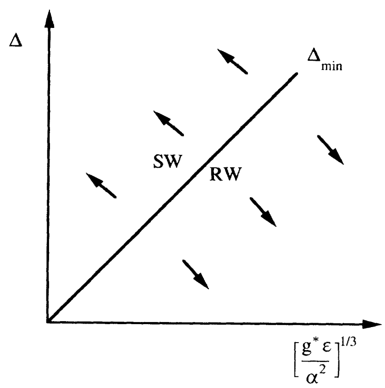

FIG. 4. Universal limit of spiral wave propagation. Spiral waves (SW) and retracting waves (RW) from, respectively, above and below this line.

Our analytical and numerical findings are summarized in Fig. 4. To conclude we note that in application of spiral-wave concepts to the heart muscle,[2,5] it has been suggested that antiarrhythmic drugs could be used to modify the properties of tissues so as to push wave propagation of neuromuscular activity in a parameter range of large R. In this range, the dangerous high-frequency spiral waves are eliminated at boundaries. For this important application, the line $\Delta_{\min}(\varepsilon)$ provides a precise knowledge of where this parameter range actually is in a two-variable model of cardiac tissues.[10] In the future, it would be important to elucidate the shape of the line corresponding in the (ε,Δ) plane to the neutral stability boundary of the "meandering" instability of spiral waves.[9,11-16] What we can conclude from the present study is that this line must lie somewhere above the line $\Delta_{\min}(\varepsilon)$.

This research is supported by Northeastern University through a grant from the Research and Scholarship Development Fund supplemented by funds from NATO.

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