Instability Criteria for the Flow of an Inviscid Incompressible Fluid

Susan Friedlander
Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60680

Misha M. Vishik
Department of Mathematics, University of Chicago, Chicago, Illinois 60637
and Institute of Mathematical Geophysics, Warshavskaya 79, K.2., 113556, Moscow, U.S.S.R.
(Received 23 January 1991)

We present a geometric estimate from below on the growth rate of a small perturbation of a three-dimensional steady flow of an ideal fluid and thus we obtain effective criteria for local instability for Euler's equations. We use these criteria to demonstrate the instability of several simple flows and to show that any flow with a hyperbolic stagnation point is unstable.

PACS numbers: 47.20.−k

In a companion paper Vishik and Friedlander[1] obtain a universal geometric estimate from below on the growth rate of a small perturbation of a three-dimensional steady flow of an inviscid incompressible fluid. In this Letter we discuss certain implications concerning instability for Euler's equations that follow from the existence of this estimate and we demonstrate that the estimate gives effective criteria for local instability. In particular, the existence of a hyperbolic stagnation point implies that the steady flow is unstable. An important feature of our approach which allows us to obtain effective criteria is that, unlike many previous approaches to hydrodynamic stability, we do not study the spectrum but rather we consider the growth rate of the relevant Green's function as $t \to \infty$.

There is a very extensive literature concerning the field of hydrodynamic stability (for references see, for example, Drazin and Reid[2]). We briefly mention some of the work whose techniques are related to those that we employ. Eckhoff and Storesletten[3] and Eckhoff[4] study the stability of azimuthal shear flows of a compressible fluid and more generally symmetric hyperbolic systems using an approach based on the generalized progressing wave expansion.[5,6] Eckhoff shows that local instability problems for hyperbolic systems can be essentially reduced to a local analysis involving ordinary differential equations (ODE) and algebraic equations only. We show that the same conclusion can be drawn for Euler's equations for an ideal fluid. These equations do not form a hyperbolic system; hence several additional technical details arise in the analysis. Bayly[7] studies the stability of quasi-two-dimensional steady flows via an analysis of a Floquet system of ODE. He shows that the Floquet exponent gives the growth rate for a family of instabilities which include the Rayleigh centrifugal instability, the Leibovich-Stewartson columnar instability, and the elliptic vortex instability. We note that the instability criteria that we present in this Letter are equivalent to those of Bayly in the particular case of quasi-two-dimensional steady flows. Lifschitz[8] uses WKB methods to construct part of the continuous spectrum for axisymmetric steady flows. Using methods inspired by magnetohydrodynamics, he obtains a necessary stability condition for a vortex ring with respect to localized three-dimensional perturbations.

Let $\mathbf{u}(\mathbf{x})$ be a steady solution of Euler's equations governing the motion in 3D of an inviscid incompressible fluid:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P\,, \qquad (1)$$

$$\nabla\cdot\mathbf{u} = 0\,. \qquad (2)$$

The 3D vector field $\mathbf{u}(\mathbf{x})$ denotes the velocity and the scalar field $P(\mathbf{x})$ denotes the pressure in the fluid. We consider the linearized Euler equations for a small perturbation velocity $\mathbf{w}(\mathbf{x},t)$. Let σ be the exact growth rate of the perturbation of the equilibrium solution. In other words, σ is the maximal real part of the spectrum of the operator

$$\mathbf{w} \to -(\mathbf{u}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{u} - \nabla Q\,, \qquad (3a)$$

$$\nabla\cdot\mathbf{w} = 0\,, \qquad (3b)$$

acting in the space of square integrable solenoidal vectors: ∇Q is chosen in a unique way to ensure that the right-hand side of (3a) has zero divergence. We discuss here only the cases of periodic boundary conditions for both $\mathbf{u}$ and $\mathbf{w}$ and the free-space problem. We assume that $\mathbf{u}$ is smooth and moreover in the free-space case we assume that derivatives of $\mathbf{u}$ are uniformly bounded.

Using techniques of WBK-type asymptotic expansions, which are analogous to those previously used in the dynamo problem, Vishik and Friedlander[1] construct the approximate Green's function for the operator (3a) and (3b). It proves useful to partition the Green's function into high-frequency and low-frequency parts. We note that for purely technical reasons it is more convenient to work with the adjoint problem which, of course, does not affect the stability results that we now present. Our approach yields the result that the growth rate σ is bounded from below by the following universal quantity of geometric nature:

$$\overline{\lim_{t\to\infty}}\,(1/t)\ln \sup_{\substack{\mathbf{x}_0,\xi_0,\mathbf{b}_0\\ |\xi_0|=1,\,\mathbf{b}_0\cdot\xi_0=0}} \|\mathbf{b}(\mathbf{x}_0,\xi_0,t)\| \le \sigma . \tag{4}$$

The vector $\mathbf{b}(\mathbf{x}_0,\xi_0,t)$, which is the first term of the amplitude of a high-frequency wavelet localized at $\mathbf{x}_0$, is obtained from the following system of ODE:

$$\dot{\mathbf{x}} = -\mathbf{u}(\mathbf{x}) , \tag{5}$$

$$\dot{\xi} = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T \xi , \tag{6}$$

$$\dot{\mathbf{b}} = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T \mathbf{b} - [(\nabla\times\mathbf{u})\times\mathbf{b}\cdot\xi]\frac{\xi}{|\xi|^2} , \tag{7}$$

with initial conditions at $t=0$

$$\mathbf{x}=\mathbf{x}_0,\quad \xi=\xi_0,\quad \mathbf{b}=\mathbf{b}_0 , \tag{8}$$

where $\xi_0\cdot\mathbf{b}_0=0$. The quantity $\xi_0/|\xi_0|$ is the direction of a spatial wave vector. The matrix $\partial\mathbf{u}/\partial\mathbf{x}$ has components $\partial u_i/\partial x_j$, $i,j=1,2,3$.

The sufficient condition for the instability given by (4) is a precise mathematical formulation of the concept of local instability widely discussed in the physical literature. We emphasize that the left-hand side of (4) involves the supremum; hence *any* Lagrangian trajectory of the flow [see (5)-(7)] could provide a positive value on the bound from below on σ and thus imply instability. It is of interest to note that criteria (4) gives a hydrodynamic analog of the criteria for a magnetic dynamo instability in an infinitely conducting fluid. The growth rate of the vector $\mathbf{b}$ satisfying (7) is an analog of the Lyapunov exponent of a dynamical system. From a mathematical point of view it appears natural to look at (7) as a linear equation over the trajectory of the system (5) and (6), which is the system in "covariations" (i.e., the evolution of a covector) for the initial dynamical system given by (5). We point out that the reverse direction of the flow appears only for the technical reasons that make it more convenient to work with the adjoint problem.

We now illustrate, via two simple examples, the fact that (4) is an effective criteria for instability in the flow of an inviscid incompressible fluid.

(1) Consider the 2D steady flow given by the stream function $\psi=\sin x_1 \sin x_2$. To demonstrate instability it is sufficient to show that the geometric quantity on the left-hand side of (4) is positive along the trajectory (5) and (6) over one stagnation point. For simplicity, we consider the origin (0,0). In this example

$$-\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\nabla\times\mathbf{u}=0$ at (0,0); hence Eq. (7) for $\mathbf{b}$ has an exponentially growing solution and the left-hand side of (4) is positive.

(2) Consider a 3D example of so-called "*ABC*" flows which are of interest in dynamo theory:

$$u_1=\cos x_2-\sin x_3,\quad u_2=\cos x_3-\sin x_1 ,$$

$$u_3=\cos x_1-\sin x_2 .$$

At the stagnation point $x_1=x_2=x_3=\pi/4$,

$$-\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and $\nabla\times\mathbf{u}=0$. It is easy to see that the above matrix has a positive eigenvalue, namely, 2; hence Eq. (7) for $\mathbf{b}$ has an exponentially growing solution.

More generally, the existence of any hyperbolic stagnation point in the steady flow $\mathbf{u}(\mathbf{x})$ gives rise to a positive lower bound for σ from the criteria (4) and hence such a flow is locally unstable. (A point $\mathbf{x}_s$ is a hyperbolic stagnation point when the spectrum of the matrix $\partial\mathbf{u}/\partial\mathbf{x}$ at $\mathbf{x}_s$ does not intersect the imaginary axis.) The following arguments justify this statement. We denote by A the matrix $(\partial\mathbf{u}/\partial\mathbf{x})^T$ at the stagnation point $\mathbf{x}_s$. For the 2D problem let $\mathbf{a}_\pm$ denote the normalized eigenvectors of A corresponding to eigenvalues $\pm\lambda$, where λ is real and positive. Then $\xi=\mathbf{a}_+e^{\lambda t}$ is a solution of (6) along the trajectory $\mathbf{x}(t)\equiv\mathbf{x}_0$. It is easy to see that

$$\mathbf{b}(t)=[\mathbf{a}_- - \mathbf{a}_+(\mathbf{a}_+\cdot\mathbf{a}_-)]e^{\lambda t}$$

satisfies Eq. (7) and hence there exists an exponentially growing solution. In the 3D problem the fact that $\mathbf{u}(\mathbf{x})$ is a steady solution of Eulers equation implies that A has as least one real positive eigenvalue λ with an eigenvector that we denote by $\mathbf{a}$. We choose $\xi=\mathbf{a}e^{\lambda t}$. It is easy to see that the matrix Λ satisfying $\Lambda\eta=-A\eta+(A\eta\cdot\mathbf{a})\mathbf{a}$ has a spectrum in the invariant subspace of vectors perpendicular to $\mathbf{a}$ given by the negative of the pair of eigenvalues of A distinct from λ. Hence Eq. (7) has an exponentially growing solution and thus by criteria (4) σ is positive. We note that the stability of certain very special flows which are exact solutions of the Navier-Stokes equations has been considered by Craik and Criminale.[9] Our results specialized to these flows are consistent with the stagnation-point instability observed by Craik and Criminale,[9] who show that under conditions this instability may persist in the presence of viscosity.

We remark that for plane-parallel shear flows the geometric quantity on the left-hand side of (4) is zero and hence this criteria says nothing about plane shear flow instability.

The results described above for a homogeneous fluid can easily be extended to the case of an inhomogeneous unbounded fluid in a gravitational field. We assume that the Boussinesq approximation is valid. The extension of

(5)-(8) in this situation is given by the following system:

$$\dot{\mathbf{x}} = -\mathbf{u}(\mathbf{x}), \tag{9}$$

$$\dot{\xi} = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T \xi, \tag{10}$$

$$\dot{\mathbf{b}} = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T \mathbf{b} - \{[(\nabla \times \mathbf{u}) \times \mathbf{b} - r\nabla\rho] \cdot \xi\} \frac{\xi}{|\xi|^2} - r\nabla\rho, \tag{11}$$

$$\dot{r} = -\mathbf{b} \cdot \nabla\Phi, \tag{12}$$

with initial conditions at $t=0$,

$$\mathbf{x} = \mathbf{x}_0, \quad \xi = \xi_0, \quad \mathbf{b} = \mathbf{b}_0, \quad r = r_0, \tag{13}$$

where $\xi_0 \cdot \mathbf{b}_0 = 0$. The symbols ρ and Φ denote the density distribution of the steady flow and the gravitational potential, respectively. We assume that $\mathbf{u}$, $\nabla\rho$, and $\nabla\Phi$ and their derivatives are bounded as $|\mathbf{x}| \to \infty$. In the instability criteria analogous to (4), $\mathbf{b}$ is replaced by the four-component vector $(\mathbf{b}, r)$ and the supremum is taken over $\mathbf{x}_0$, ξ_0, $\mathbf{b}_0$, r_0; $\xi_0 \cdot \mathbf{b}_0 = 0$.

S. F. acknowledges the Institut des Hautes Etudes Scientifiques (IHES) for its hospitality and support and acknowledges NSF Grant No. DMS 9000137. M. V. is grateful to the University of Chicago for its extremely kind hospitality. We thank A. Lifschitz for bringing Refs. 3, 4, and 9 to our attention.

[1]M. M. Vishik and S. Friedlander, "Dynamo Theory Methods for Hydrodynamic Stability" (to be published).

[2]P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge Univ. Press, Cambridge, 1981).

[3]K. S. Eckhoff and L. Storesletten, J. Fluid Mech. **99**, 433 (1980).

[4]K. S. Eckhoff, J. Differential Equations **40**, 94 (1981).

[5]F. G. Friedlander, *Sound Pulses* (Cambridge Univ. Press, Cambridge, 1958).

[6]D. Ludwig, Commun. Pure Appl. Math. **13**, 473 (1960).

[7]B. J. Bayly, Phys. Fluids **31**, 56 (1988).

[8]A. Lifschitz, Phys. Lett. A **152**, 199 (1991).

[9]A. D. D. Craik and W. O. Criminale, Proc. Roy. Soc. London A **406**, 13 (1986).