## Directed Waves in Random Media

Recently, Feng, Golubović, and Zhang[1] (FGZ) studied propagation of directed waves in random media. By analogy to directed polymers (DP),[2] they suggested that transverse displacements of light paths grow superdiffusively, and confirmed this by numerical simulations. Here we show the statistical similarity of the FGZ results to a variant of the DP problem,[3] and also point out how unitarity constraints modify these results.

Consider $\Psi(\mathbf{x},t)$ evolving as

$$\partial\Psi/\partial t = [\gamma\nabla^2 - \mu(\mathbf{x},t)]\Psi ,$$

where $\mu(\mathbf{x},t)$ is a random function, uncorrelated at different points $(\mathbf{x},t)$. We distinguish between four cases: (1) $\gamma$ and $\mu$ are both real, as in the DP problem[2] in a random medium. (2) $\gamma$ is real and $\mu$ is imaginary, relevant to electron tunneling under a random barrier.[3] (3) $\gamma$ and $\mu$ are both imaginary, as in the parabolic wave equation studied by FGZ. (4) $\gamma$ is imaginary and $\mu$ is real, which will not be discussed here. In (3) the evolution is unitary in that $\Psi(\mathbf{x},t)$ stays normalized at all $t$. Using the path-integral solution

$$\Psi(\mathbf{x},t) = \int_{(0,0)}^{(\mathbf{x},t)} \mathcal{D}\mathbf{x}(\tau) \exp\left\{ -\int d\tau \left[ \frac{\dot{\mathbf{x}}^2}{2\gamma} + \mu(\mathbf{x},\tau) \right] \right\},$$

we examine the statistical fluctuations of $\Psi(\mathbf{x},t)$. Replicating the above, we see that in (2) and (3), only moments such as $\langle(\Psi^*\Psi)^n\rangle$ survive averaging over the imaginary random potential. Contributions to such moments come from $n$ pairs of coupled time-reversed paths.[3,4] The paired paths in (2) are subject to a line tension $\gamma_e^{-1} = 2\gamma^{-1}$, while in (3) the apparent line tension is zero. A vanishing line tension leads to wildly fluctuating (nonlocal) paths, probably invalidating the approximations leading to the parabolic equation. In fact, any local discretization or constraint on paths generates an entropic line tension $\gamma_e^{-1}$, and we focus on such situations in this Comment.

If intersections between paired paths are ignored, the number of possible pairings leads to the classical result[4] $\langle(\Psi^*\Psi)^n\rangle = n!\langle\Psi^*\Psi\rangle^n$; i.e., intensity fluctuations satisfy a Poisson distribution. However, in (2) every time two paired paths intersect they can exchange partners, leading to a statistical attraction of 2 per crossing.[3] If $d$, the dimension of $\mathbf{x}$, is less than or equal to two, intersections are frequent and change the statistics, leading to $\langle(\Psi^*\Psi)^n\rangle \approx \langle\Psi^*\Psi\rangle^n e^{an^p t}$ for large $t$. As such attraction is also directly generated by averaging in (1),[2] both (1) and (2) are characterized by a log-normal distribution of intensity fluctuations, in which the variance of $\ln(\Psi^*\Psi)$ grows as $t^{1/p}$. (Results for DP indicate $p=3$ in $d=1$, and $p\approx 5$ in $d=2$.) Does this distribution also apply to (3), as suggested by a similar path-integral argument? A power-law growth of moments with $t$ is in fact inconsistent with unitarity constraints, which on a lattice lead to $(zt)^{-d} < |\Psi(0,t)|^2 < 1$. Thus a strictly unitary time evolution implies $a=0$.

We performed numerical simulations, first using the discretization of FGZ in $d=1$, *which does not preserve unitarity.* With a disorder strength $v=5$, we averaged over 1000 realizations of randomness. The intensity $\Psi^*\Psi$ was indeed log-normally distributed, and its variance grew as $t^{1/p}$, with $1/p = 0.35 \pm 0.04$, slightly larger than, but consistent with, $p=3$. We also found that the beam positions $\langle[x(t)]^2_{\text{ave}}\rangle$ and $\langle[x(t)^2]\rangle_{\text{ave}}$, scale as $t^{2\nu}$ with $\nu = 0.67 \pm 0.04$. The exponent values are indeed similar to previous results[3] for (2). We thus conclude that if unitarity is not an essential constraint, the statistical fluctuations in $\Psi$ are similar in cases (1), (2), and (3). We next repeated the simulations using a Cayley-form discretization *which preserves unitarity,* and multiplying $\Psi(x,t)$ with a random phase $e^{i\phi(x,t)}$ at each step. No anomalous fluctuations in logarithmic intensity were observed, and the statistics of $\Psi^*\Psi$ appeared to be Poisson. The beam positions $\langle[x(t)]^2_{\text{ave}}\rangle$ and $\langle[x(t)^2]_{\text{ave}}\rangle$ scaled as $t^{0.60\pm0.08}$ and $t^{1.00\pm0.08}$, respectively. In fact, $\langle[x(t)^2]_{\text{ave}}\rangle = \gamma_e^{-1}t$ is another consequence of unitarity.

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