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Correlation Functions in Liouville Theory

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We show that after integrating over the zero mode in Liouville correlation functions, the remaining functional integral resembles a free theory and may be evaluated by formally continuing the central charge. We apply this technique to the unitary minimal models coupled to gravity on the sphere, computing a number of three-point functions. After taking into account the normalizations of operators and the functional integral, we find exact agreement between the Liouville three-point functions and the results from matrix models.

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Given the relative ease with which two-dimensional quantum gravity may be solved with matrix models,[1] it is somewhat surprising that the continuum formulation, which ostensibly describes the same physics, has proved so difficult. Although scaling weights may be extracted rather easily,[2,3] the numerical coefficients appearing in correlation functions require more work. Some progress was recently made in Ref. 4 where the p, q, and radius dependences of the genus-one partition function for Liouville theory coupled to minimal models and a compactified boson were determined. An important step in the argument was to treat the zero mode of the Liouville field ϕ separately from the remaining modes (see also Refs. 5 and 6).

In this paper we show that after integrating over the zero mode of ϕ in Liouville correlation functions, the remaining functional integral resembles a free-field correlator and may be computed by formally continuing the value of the central charge. We apply this technique to the unitary minimal models coupled to gravity on the sphere. We compute the partition function and two- and three-point functions of dressed operators on the diagonal of the Kac table. For three-point functions, the algebra is quite involved due to the complexity of operator-product coefficients in the minimal models.[7] For this reason, we complete the computation only for a subset of three-point functions. This restriction, however, is only to simplify the operator-product coefficients. There is no exceptional simplification in the Liouville sector and the general three-point function may be computed by the same techniques. Using the partition function and two-point functions to fix normalizations, we find exact numerical agreement between the Liouville three-point functions and the results from matrix models.[8]

(1) Liouville theory.—We start with Liouville theory in the conformal gauge.[3,9] The action is given by

$$S_L=\int \frac{1}{2\pi}\partial\phi\,\bar\partial\phi-\frac{1}{8\pi}Q\sqrt{\hat g}\hat R\phi+\frac{\mu}{\pi}\sqrt{\hat g}e^{\alpha\phi}\,,$$

where

$$Q=\sqrt{(25-c)/3},\quad \alpha=-Q/2+\tfrac{1}{2}(Q^2-8)^{1/2}.$$

We have fixed a background metric $\hat g$, with curvature $\hat R$ normalized by

$$\frac{1}{8\pi}\int\sqrt{\hat g}\hat R=1-h$$

on a genus-h surface. As discussed in Refs. 3 and 9, the measure for ϕ is translation invariant.

Consider correlations of the form

$$\left\langle\prod_{i=1}^{n}e^{\beta_i\phi(z_i)}\right\rangle=\int\mathcal{D}\phi\,e^{-S_L}\prod_{i=1}^{n}e^{\beta_i\phi(z_i)}.$$

In order to evaluate the functional integral, we first integrate over the zero mode of ϕ. Note that the zero-mode integral may diverge, depending on c, β_i, and h. In this case the integral will be defined by analytic continuation in c.

We define

$$\phi(z)\equiv\tilde\phi(z)+\phi_0\,,\tag{1.1}$$

where ϕ_0 denotes the kernel of the scalar Laplacian (constants) and $\tilde{\phi}(z)$ the space of functions orthogonal to the kernel

$$\int \sqrt{\hat{g}}\tilde{\phi}=0 .$$

Integrating over ϕ_0 we find

$$\left\langle\prod_{i=1}^{n} e^{\beta_i \phi(z_i)}\right\rangle=\left(\frac{\mu}{\pi}\right)^s \frac{\Gamma(-s)}{\alpha} \int \mathcal{D}\tilde{\phi}\, e^{-S_0}\left(\int \sqrt{\hat{g}}\, e^{\alpha\tilde{\phi}}\right)^s \prod_{i=1}^{n} e^{\beta_i \tilde{\phi}(z_i)} \tag{1.2}$$

where

$$\begin{aligned} S_0&=\int \frac{1}{2\pi}\partial\tilde{\phi}\bar{\partial}\tilde{\phi}-\frac{1}{8\pi}Q\sqrt{\hat{g}}\hat{R}\tilde{\phi} ,\\ s&=-\frac{Q}{\alpha}(1-h)-\sum_{i=1}^{n}\frac{\beta_i}{\alpha} .\end{aligned} \tag{1.3}$$

Assuming for the moment that s is an integer, then the remaining functional integral in (1.2) is a free-field correlator. (A similar observation was made in Ref. 6.) Recall that in a correlation function of vertex operators in a scalar field theory with total charge (momentum) zero, the functional integral is independent of the zero mode and a splitting as in (1.1) must be made to remove the divergence from the zero-mode integration. The end result is that the Green's function used to evaluate (1.2) is orthogonal to the zero mode (see, for example, Ref. 10).

Unfortunately, in general, s will not be a positive integer. In this case, we will first evaluate the functional integral in (1.2) for c such that s is a positive integer and then formally continue c back to its original value. We should emphasize that this continuation in c is not an analytic continuation. We first determine the general form of (1.2) for a discrete set of values for c, corresponding to integer s, and then formally evaluate this function on a value of c outside of this set, corresponding to noninteger s. Such a continuation is of course ambiguous. We will return to this point at the end of the paper.

(2) Minimal models.—We now use the above technique to compute some correlation functions in Liouville theory coupled to unitary minimal models, with

$$c=1-6/q(q+1) . \tag{2.1}$$

Of course, there is an ambiguity in the normalizations of operators and the measure of the functional integral. Therefore, in order to compare with matrix models, we must compute appropriate ratios of correlators. The simplest nontrivial example is

$$\frac{\langle A_i A_j A_k\rangle^2 Z}{\langle A_i A_i\rangle\langle A_j A_j\rangle\langle A_k A_k\rangle} , \tag{2.2}$$

where A_i, A_j, and A_k are arbitrary operators and Z denotes the partition function in Liouville theory (the universal part of the free energy in matrix models). We will compute (2.2) for operators on the diagonal of the Kac table, with appropriate gravitational dressing.[3] We will also restrict to genus zero.

The dressed fields are

$$A_i \equiv \psi_{r_i r_i} e^{\beta_i \phi} ,$$

where $\psi_{r_i r_i}$ is a primary field of dimension

$$h_{r_i r_i}=(r_i^2-1)/4q(q+1)$$

and

$$\beta_i=(r_i-2q-1)/\sqrt{2q(q+1)} .$$

The correlator of interest is

$$\langle A_{r_1}A_{r_2}A_{r_3}\rangle=\int \prod_{i=1}^{3}\sqrt{\hat{g}(z_i)}\,d^2z_i\left\langle\prod_{i=1}^{3}\psi_{r_i r_i}(z_i)\right\rangle\left\langle\prod_{i=1}^{3}e^{\beta_i\phi(z_i)}\right\rangle .$$

Two-point functions may be obtained from $\langle A_1A_2A_3\rangle$ by setting $r_2=r_1$, $r_3=1$, and integrating once with respect to μ (and similarly for the partition function). We have suppressed the ghost fields and fix the SL(2,$\mathbb{C}$) invariance by hand, dropping the integrals over z_i and setting $z_1=0$, $z_2=1$, and $z_3=\infty$. After evaluating the matter three-point function[11] and integrating over the Liouville mode as described in the previous section, we find

$$\langle A_{r_1}A_{r_2}A_{r_3}\rangle=D^{(r_3r_3)}_{(r_1r_1)(r_2r_2)}\left(\frac{\mu}{\pi}\right)^s\frac{\Gamma(-s)}{\alpha}\int\prod_{i=1}^{s}d^2w_i\prod_{i<j}|w_i-w_j|^{-2\alpha^2}\prod_{i=1}^{s}|w_i|^{-2\alpha\beta_1}|w_i-1|^{-2\alpha\beta_2} , \tag{2.3}$$

where $D^{(r_3r_3)}_{(r_1r_1)(r_2r_2)}$ is the operator-product coefficient that appears in the matter three-point function and s is defined in (1.3). The above multiple integral is evaluated in Ref. 12 (formula B.9). This gives

$$\langle A_{r_1}A_{r_2}A_{r_3}\rangle=D^{(r_3r_3)}_{(r_1r_1)(r_2r_2)}\left(\frac{\mu}{\pi}\right)^s\frac{\Gamma(-s)}{\alpha}\Gamma(s+1)\pi^s\frac{\Gamma(1+\rho')^s}{\Gamma(-\rho')^s}\prod_{i=1}^{s}\left[\frac{\Gamma(-i\rho')}{\Gamma(1+i\rho')}\prod_{j=1}^{3}\frac{\Gamma(r_j-(r_j+i)\rho')}{\Gamma(1-r_j+(r_j+i)\rho')}\right] , \tag{2.4}$$

where

$$\rho'=\frac{q}{q+1}=\frac{r_1+r_2+r_3-1}{2s+1+r_1+r_2+r_3} .$$

The variable q is defined for arbitrary c by (2.1). Unfortunately, we cannot yet restore q to a positive integer (corresponding to noninteger s), since in (2.4) s is the upper bound of the index i. Using the identities[13]

$$\Gamma(nx)=(2\pi)^{(1-n)/2}n^{nx-1/2}\prod_{k=0}^{n-1}\Gamma(x+k/n)\,,\quad \Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)\,,$$

one may derive

$$\begin{aligned}\prod_{i=1}^{s}\frac{\Gamma(r_j-(r_j+i)\rho')}{\Gamma(1-r_j+(r_j+i)\rho')}&=\rho^{n^2\rho+n(\rho+1)-2r_jn(\rho-1)-2}r_j^2(1-\rho)^2\\&\quad\times\Gamma(r_j(1-\rho))^2S(r_j\rho)\prod_{i=1}^{n}\frac{S(i\rho')}{S(i\rho)}\prod_{i=1}^{r_j}\frac{S(i\rho')S((n+1-i)\rho)}{S(i\rho)S((n+1-i)\rho')}\\&\quad\times\prod_{i=0}^{n}\frac{\Gamma(1+n+r_i-(i-r_j)\rho')^2}{\Gamma(1+r_j-(r_j-i)\rho)^2}\,,\\ \prod_{i=1}^{s}\frac{\Gamma(-i\rho')}{\Gamma(1+i\rho')}&=\rho^{n^2\rho+n(\rho+1)-2}S(n\rho)\prod_{i=1}^{n}\frac{S(i\rho')}{S(i\rho)}\prod_{i=0}^{n}\frac{\Gamma(1+n-i\rho')^2}{\Gamma(1+i\rho)^2}\,,\end{aligned}\tag{2.5}$$

where

$$\rho=\frac{1}{\rho'},\quad n=\frac{1}{2}\left[\sum_{i=1}^{3}r_i-1\right],\quad S(x)\equiv\frac{\sin(\pi x)}{\pi}\,.$$

Note that on the right-hand side of the expressions in (2.5), s only appears through ρ and ρ'. Thus, after substituting (2.5) in (2.4), we may take q to be a positive integer.

We first consider the special case $r_1=r_2\equiv r$, $r_3=1$, with $D^{(11)}_{(rr)(rr)}=1$. We find from (2.4) and (2.5)

$$\langle A_rA_rA_1\rangle=r^2(1-\rho)^2\rho^{-4}\frac{S(r/q)}{S(r/(q+1))}\frac{\Gamma(-r/q)^2}{\Gamma(r/(q+1))^2}\left[\frac{\Gamma(\rho')}{\Gamma(1-\rho')}\mu\right]^{r/q}\frac{1}{\mu}\,.$$

Using

$$\langle A_rA_rA_1\rangle=\frac{d}{d\mu}\langle A_rA_r\rangle\,,$$

we thus find for the two-point function

$$\langle A_rA_r\rangle=q(1-\rho)^2\rho^{-4}r\frac{S(r/q)}{S(r/(q+1))}\frac{\Gamma(-r/q)^2}{\Gamma(r/(q+1))^2}\left[\frac{\Gamma(\rho')}{\Gamma(1-\rho')}\mu\right]^{r/q}.\tag{2.6}$$

We may also determine the partition function by setting $r=1$ in (2.6) and integrating

$$\langle A_1A_1\rangle=\frac{d^2}{d\mu^2}Z\,.$$

After some simple algebra, Z may be written as

$$Z=\frac{(1-\rho)^2q}{\rho^5(1+\rho)\Gamma(\rho)^2\Gamma(\rho')^2S(1/q)S(1/(q+1))}\left[\frac{\Gamma(\rho')}{\Gamma(1-\rho')}\mu\right]^{2+1/q}.\tag{2.7}$$

The operator-product coefficients $D^{(r_3r_3)}_{(r_1r_1)(r_2r_2)}$ have been computed in Ref. 7. Unfortunately, the expressions are quite complex. For this reason we specialize to the case

$$r_1+r_2+r_3=2q-1\,,\tag{2.8}$$

where we have been able to simplify the operator-product coefficients to the more manageable form

$$\begin{aligned}(D^{(r_3r_3)}_{(r_1r_1)(r_2r_2)})^2&=S(1/q)S(1/(q+1))^3\frac{\Gamma(\rho)^2\Gamma(\rho')^6}{(1-\rho)^8}\\&\quad\times\left[\prod_{i=1}^{3}r_i^2S(r_i/q)S(r_i/(q+1))\right.\\&\qquad\left.\times S((1+r_i)/(q+1))^2\Gamma(-r_i/q)^2\Gamma(r_i/(q+1))^2\Gamma((1+r_i)/(q+1))^4\right]^{-1}.\end{aligned}\tag{2.9}$$

We emphasize, however, that condition (2.8) is imposed only to simplify the matter three-point function and does not imply any special simplification in the Liouville sector. The general three-point function may be treated by the same techniques and it is only algebraic complexity that prevents us from discussing it here.

Assuming (2.8) and using (2.4), (2.5), and (2.9), we find for the square of the three-point function

$$\langle A_{r_1}A_{r_2}A_{r_3}\rangle^2=\frac{(1-\rho)^4}{\rho^8}S(1/q)S(1/(q+1))\Gamma(\rho)^2\Gamma(\rho')^2 \times\left[\prod_{i=1}^{3}r_i^2\frac{S(r_i/q)\Gamma(-r_i/q)^2}{S(r_i/(q+1))\Gamma(r_i/(q+1))^2}\right]\left[\frac{\Gamma(\rho')}{\Gamma(1-\rho')}\mu\right]^{-2/q}. \tag{2.10}$$

Combining (2.6), (2.7), and (2.10) we find

$$\frac{\langle A_iA_jA_k\rangle^2 Z}{\langle A_iA_i\rangle\langle A_jA_j\rangle\langle A_kA_k\rangle}=\frac{r_1r_2r_3}{(q+1)(2q+1)}. \tag{2.11}$$

For comparison, we list the results from matrix models,[8]

$$\langle A_rA_r\rangle_{\text{mat mod}}=2r\left[\frac{\mu}{q+1}\right]^{r/q},$$
$$\langle A_{r_1}A_{r_2}A_{r_3}\rangle_{\text{mat mod}}=\frac{2}{q}r_1r_2r_3\left[\frac{\mu}{q+1}\right]^{(r_1+r_2+r_3-1)/2q-1}, \tag{2.12}$$
$$Z_{\text{mat mod}}=\frac{2q^2}{(q+1)(2q+1)}\left[\frac{\mu}{q+1}\right]^{2+1/q}.$$

Taking ratios of correlators we see that the matrix models also give (2.11).

The prescription we have used to continue from integer to noninteger s for three-point functions is given in (2.5). In other words, the left-hand side of (2.5) is defined only for integer s, whereas the right-hand side is defined for all s. These expressions were obtained by straightforward algebraic manipulation and were the only forms we found that allowed such a continuation. While we agree that one may artificially add terms which vanish for integer s but are nonzero for noninteger s, we have not seen such expressions appear "naturally." Finally, the exact agreement between our computations and the results from matrix models suggests that there is a rigorous justification for our continuation; however, we have not yet found one.

Conclusion.— We have shown that it is possible to compute correlation functions in continuum two-dimensional quantum gravity by formally continuing the value of the central charge. For unitary minimal models, the correlators we compute appear to be quite different from the matrix-model results. However, after taking into account the normalizations of the partition function and operators, we are in exact agreement with matrix models.

We feel that this agreement justifies our approach. Unfortunately, we have so far not been able to extend the technique beyond calculating certain three-point functions in Liouville theory coupled to minimal models as described above. We are thus unable to make contact with, for example, the weak-coupling analysis of Ref. 5 in which the zero mode is also treated separately from the rest of the Liouville field.

Our approach suffers from a number of drawbacks. The computations involved quite a bit of tedious algebra. However, even so, our work was made easier because the difficult multiple integrals had already been evaluated in Ref. 12. In order to compute more general correlation functions, one would first have to derive generalizations of the formulas in Ref. 12. The problems get worse at higher genus where the Green's functions are more complicated and integrals over moduli appear. Nevertheless, it may be possible to use the above techniques to gain some insight into quantum gravity coupled to matter with $c \geq 1$.

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