Scaling in Open Dissipative Systems

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The scaling behavior of open dissipative systems including the Kim-Kosterlitz exponents for interfacial growth, and the current fluctuations in a flowing sandpile are derived using a generalization of the arguments applied by Kolmogorov to the inertial range of turbulence. The approach may be considered a nonequilibrium equivalent to Flory theory.

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The scaling behavior of open dissipative systems far from thermal equilibrium such as interfacial growth,[1] self-organized criticality (SOC),[2] and $1/f$ noise involve the nonlinear collective interactions of numerous degrees of freedom. One approach to studying this behavior is by deriving Langevin-type equations which are assumed to incorporate the physics. Typical of such equations is the Kardar-Parisi-Zhang (KPZ) equation[3] for the height fluctuations $h(\mathbf{r},t)$ in an interface growing with a velocity λ normal to the interface

$$\partial h/\partial t = \nu\nabla^2 h + (\lambda/2)(\nabla h)^2 + \eta(\mathbf{r},t) , \tag{1}$$

where

$$\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = D\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') .$$

Other examples would include the Hwa-Kardar (HK) equation[4] for fluctuations in a flowing sandpile, and the Sun-Guo-Grant (SGG) equation[5] for the surface height of a driven interface with a conservation law.

These nonlinear equations are, in general, insoluble. Therefore, most of the efforts in the past have been focused on determining the scaling behavior of the fluctuations using a dynamic renormalization-group (RG) approach or by direct numerical solutions of these equations.[6] The dynamic RG has had only a limited success in the study of dissipative systems because there is no Hamiltonian formulation for nonequilibrium processes and in most cases RG equations cannot be formulated or solved. Numerical solutions, on the other hand, are of practical importance, but can only give approximate values of the scaling exponents. But since numerical results are only approximate and cannot be used to determine universality and crossover behavior, they do not provide physical insight into these processes. In addition, unlike the scaling behavior at a critical point which is described by a single exponent, fluctuations in dynamical systems can have different scaling behavior depending on the length scale. Thus, it would be useful to have an approach that could be readily applied to Langevin-type equations and which could be used to determine the exponents in any dimensions for different scaling regimes.

In this Letter we propose a new approach for studying the scaling behavior of Langevin-type equations for dissipative dynamical systems. Our approach is similar in spirit to the scaling arguments used by Kolmogorov in the analysis of fully developed turbulence and is based on the analogy between Langevin-type equations and the forced Navier-Stokes equation. We show that this approach can be applied to derive not only the critical exponents in any dimension, but also the fluctuation amplitudes, critical dimensions, and regimes of validity, where various exponents may be observed. This approach may be considered a nonequilibrium equivalent of the Flory theory for equilibrium scaling. We demonstrate the approach by several examples; some such as how the Kim-Kosterlitz exponents may be derived from the KPZ equation are new, others such as the dynamic and roughening exponents for the HK equations and the current fluctuations in SOC are rederived to show the generality of the approach. The previous examples involve white noise, and therefore we also consider our arguments in the presence of colored noise and quenched randomness.

Let us begin by noting that if all the transport coefficients such as ν, λ, and D in the KPZ equation depend on purely microscopic length scales a, then on scales $l \gg a$ these equations describe the macroscopic behavior in the same manner as the Navier-Stokes equation,

$$\partial \mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v} = -(1/\rho)\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}(\mathbf{r},t) , \tag{2}$$

describes turbulent flow.[7] Here $\nabla\cdot\mathbf{v}=0$ for the velocity field of an incompressible fluid, with $\langle \mathbf{f}(\mathbf{r},t)\cdot\mathbf{f}(\mathbf{r}',t')\rangle = \epsilon\delta(t-t')$ being the noise correlation function if we are considering the inertial range of turbulence on scales below the stirring length scale l_{out} at which a mean energy input per unit mass per unit time ϵ is being pumped into the fluid. Indeed, by substituting $\mathbf{v} = -\nabla h$ and $\mathbf{f} = -\nabla\eta$, it is possible to transform Eq. (1) into the Burgers equation[8,9] $\partial\mathbf{v}/\partial t + \lambda\mathbf{v}\cdot\nabla\mathbf{v} = \nu\nabla^2\mathbf{v} + \mathbf{f}(\mathbf{r},t)$, which makes the similarity all the more striking. This analogy extends to the visual domain; for instance, the images of turbulent boundary layers near walls[10] appear remarkably similar in structure to computer simulations of interfacial growth.

This suggests that modified and generalized versions of the type of scaling arguments introduced by Kolmogorov[11] in the context of the inertial range of three-dimensional homogeneous turbulence and extended to the study of fractally homogeneous turbulence[12] by Hentschel and Procaccia[13,14] might be useful as a method to identify the different scaling regimes observable in open dissipative systems.

Basically for any equation such as Eq. (1) to show scaling each separate term (including the noise), when coarse-grained over length scales l, must be of the same order of magnitude or negligible. Only under these circumstances can scaling behavior arise. The validity of a scaling regime can be then found in a self-consistent manner from the region of length scales over which the intrinsic assumptions apply. The art, as in Flory theory, lies in estimating the magnitude of individual terms, especially as, in general, we are dealing with self-affine and/or anisotropic systems, which introduces several length scales into the estimate. We illustrate the approach by a few examples. Clearly, the range of applicability of this approach goes well beyond these examples.

Interface growth, KPZ equation.—The first open system we examine is noise-driven surface growth, which is expected to be described by the KPZ equation (1). Recent interest in the growth of surfaces has resulted in the recognition that several models for such growth on interfaces of dimensions L^{d-1} all obey a form of dynamic scaling:[15]

$$W(L,t)\sim\begin{cases}t^{\beta} \text{ if } t\ll t_L\,,\\ L^{\alpha} \text{ if } t\gg t_L\,,\end{cases} \tag{3}$$

where $W(L,t)$ is a measure of the width of the surface at time t on a length scale L, and $t_L\sim L^z$ with $z=\alpha/\beta$. The results of dynamic RG and various growth models[1,15] are consistent with each other in $d=2$ giving $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$. There are not exact results in $d>2$, but on the basis of numerical evidence alone, Kim and Kosterlitz[16] have suggested $\alpha=2/(d+2)$ and $\beta=1/(d+1)$. In $d=3$ the Kim-Kosterlitz exponents are close to the values obtained by numerical solution of the KPZ equation.[6]

We assume that at long times $t\gg t_l$, and averaged over length scales l, the typical magnitude of the fluctuations in the interfacial height scale as $\langle[h(\mathbf{r}+l,t)-h(\mathbf{r},t)]^2\rangle_l\sim h_l^2$, and that at long times these fluctuations last for times of the order t_l. Then, apart from the noise, for times $t\gg t_l$ and averaged over scales l, the various terms in the KPZ equation may be estimated as $\langle|\partial h/\partial t|\rangle_l\sim h_l/t_l$, $\nu\langle|\nabla^2 h|\rangle_l\sim \nu h_l/l^2$, and $(\lambda/2)\times\langle(\nabla h)^2\rangle_l\sim\lambda h_l^2/l^2$.

To proceed further we need to estimate the average noise on these length and time scales. For white noise we estimate its mean-square fluctuations on length scales l and time scales t_l as $\eta_l\sim(D/S_l t_l)^{1/2}$, where S_l is the average surface area of the interface on length scales l. This is a simple consequence of adding uncorrelated random variables. We estimate the surface area of the growth on length scales l as $S_l\sim(h_l^2+l^2)^{(d-1)/2}$, and consequently for smooth surfaces $\eta_l\sim(D/l^{d-1}t_l)^{1/2}$, while for rough surfaces $\eta_l\sim(D/h_l^{d-1}t_l)^{1/2}$.

To derive the Kim-Kosterlitz exponents we assume that at sufficiently large length scales $l\gg l_{\rm in}$, the nonlinear term in the KPZ equation will dominate the surface diffusion. The regime where this assumption is valid is defined by $h_l\gg\nu/\lambda$. Equating the $\partial h/\partial t$ term with the nonlinear term implies that a typical fluctuation lasts for time $t_l\sim l^2/\lambda h_l$. The scaling behavior of these two terms implies $\alpha+z=2$.

Equating our estimate for the noise fluctuation in a rough interface $h_l\gg l$ (a condition yielding an outer length scale $l_{\rm out}$) to the inertial term then yields

$$h_l\sim(D/\lambda)^{1/(d+2)}l^{2/(d+2)}\,, \tag{4}$$

and consequently $\alpha=2/(d+2)$ in this regime. The inner length scale $l_{\rm in}\sim(\nu^{d+2}/D\lambda^{d+1})^{1/2}$ can now be found by inserting Eq. (4) into the self-consistency condition $h_l\gg\nu/\lambda$. We can find the scaling behavior of h_t with time t at short times be reexpressing h_l in terms of t_l and assuming scaling is valid for $t\ll t_l$ with the result

$$h_t\sim D^{1/(d+1)}t^{1/(d+1)}\,, \tag{5}$$

and therefore $\beta=1/(d+1)$.

Thus, we have derived theoretically the expressions conjectured by Kim and Kosterlitz[16] for α and β on the basis of numerical results, as well as their fluctuation amplitudes and inner length scale $l_{\rm in}$. The outer length scale can be found by substituting Eq. (4) into the criterion for the existence of a rough interface yielding $l_{\rm out}\sim(D/\lambda)^{1/d}$, and this expression implies that we may expect to observe the exponents only in models in the strong-coupling limit where the dimensionless parameter $\epsilon\equiv\lambda^{d-1}D/\nu^d\gg1$ which is analogous to the Reynolds number describing hydrodynamic turbulence.

If $\epsilon\ll1$, it is also possible to find a regime where the Edwards-Wilkinson exponents[17,18] are valid, with

$$h_l\sim(D/\nu)^{1/2}l^{(3-d)/2}\,, \tag{6}$$

and consequently $\alpha=(3-d)/2$, while

$$h_t\sim(D^2\nu^{1-d})^{1/4}t^{(3-d)/4}\,, \tag{7}$$

and thus $\beta=(3-d)/4$. These exponents are perhaps less interesting than the Kim-Kosterlitz exponents as they can be derived exactly from the linearized version of the KPZ equation; however, they also show the generality of the scaling arguments applied and can also be used to study crossover between regimes.

Medina *et al.*[9] and Zhang[19] consider a generalization of the KPZ equation for interfacial growth in which the noise in Eq. (1) instead of being δ correlated in space has the correlation function

$$\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle=D'|\mathbf{r}-\mathbf{r}'|^{2\rho-(d-1)}\delta(t-t')\,.$$

We expect the effect of long-range correlations in the noise to change our estimate of the noise fluctuation averaged over length scales l and time scales t_l into $\eta_l \sim (D' h_l^{2\rho-(d-1)}/t_l)^{1/2}$. As all other relationships remain unchanged, the behavior of h_l and h_t can immediately be found with the result

$$h_l \sim (D'/\lambda)^{1/(2+d-2\rho)} l^{2/(2+d-2\rho)}, \qquad (8)$$

and therefore $\alpha = 2/(2+d-2\rho)$, while

$$h_t \sim (D't)^{1/(1+d-2\rho)}, \qquad (9)$$

and thus $\beta = 1/(1+d-2\rho)$. Of course, these results are only valid for $\alpha < 1$ and so long as the noise correlation function decays, and, therefore, only for $\rho < \rho_{\max} = (d-1)/2$. Also these exponents can only be expected in the case of pure colored noise: Any white-noise contribution will lead to crossover effects.

Surface growth with conservation law.—The KPZ equation does not conserve the total volume of the interface in the absence of external forcing. In order to study this conservation law on interfacial growth Sun, Guo, and Grant[5] used the dynamic renormalization group to study the nonlinear Langevin equation

$$\partial h/\partial t = -\nabla^2[\nu \nabla^2 h + (\lambda/2)(\nabla h)^2] + \eta(\mathbf{r},t), \qquad (10)$$

where

$$\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = -2D\nabla^2 \delta(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

Again neglecting the diffusion term as small at large enough length scales and equating our estimate for the time variation in the height fluctuations $\langle |\partial h/\partial t|\rangle_l \sim h_l/t_l$ to our estimate for the nonlinear term $(\lambda/2)\nabla^2(\nabla h)^2 \sim \lambda h_l^2/l^4$ in Eq. (10) yields the identity $\alpha + z = 4$, while the estimate for the noise $(D/l^{d+1}t_l)^{1/2}$ yields

$$h_l \sim (D/\lambda)^{1/3} l^{(3-d)/3}, \qquad (11)$$

and consequently $\alpha = (3-d)/3$ with $d_c = 3$, and

$$t_l \sim (D\lambda^2)^{-1/3} l^{(9+d)/3}, \qquad (12)$$

and consequently $z = (9+d)/3$ in this regime. These results are in agreement with SGG.[5]

Self-organized criticality.—As another example, consider the current fluctuations and avalanches in a flowing sandpile. Hwa and Kardar[4] introduced another driven Langevin equation incorporating the symmetries and conservation laws of the Bak, Tang, and Weisenfeld[2] discrete sandpile model for self-organized criticality. As the sand has a macroscopic flow direction, the resulting equation,

$$\partial h/\partial t = \nu_\parallel \partial_\parallel^2 h + \nu_\perp \nabla_\perp^2 h - (\lambda/2)\,\partial_\parallel h^2 + \eta(\mathbf{r},t), \qquad (13)$$

is anisotropic; the noise due to the added sand grains is taken to be white,

$$\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = 2D\delta(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

Again in this model the dynamic exponent z, and the roughening exponent α, can be found. But in this case because the problem is both self-affine and anisotropic, there are three length scales rather than two involved in the analysis; thus if l is a typical scale to be studied along the flow direction, then associated with this parallel length scale is a transverse length scale $l_\perp \sim l^\zeta$, where ζ is the spatial anisotropy exponent, and again we call a fluctuation in the sandpile height on these scales h_l. Using these three length scales, the various terms in Eq. (13) can be estimated as $\langle|\partial h/\partial t|\rangle_l \sim h_l/t_l$, $\nu_\parallel \langle|\partial_\parallel^2 h|\rangle_l \sim \nu_\parallel h_l/l^2$, $\nu_\perp\langle|\nabla_\perp^2 h|\rangle_l \sim \nu_\perp h_l/l_\perp^2$, and $(\lambda/2)\langle|\partial_\parallel h^2|\rangle_l \sim \lambda h_l^2/l$. Estimating the area of the sandpile surface on length scales l to be $S_l \sim l l_\perp^{d-2}$ gives a noise estimate $\eta_l \sim (D/l l_\perp^{d-2} t_l)^{1/2}$. Neglecting at large length scales the parallel component of sandpile relaxation through surface tension compared to the parallel nonlinear transport term yields the exponent equalities

$$\alpha - z = \alpha - 2\zeta = 2\alpha - 1 = -[z+1+(d-2)\zeta]/2,$$

which can be reexpressed as $z = 6/(8-d)$, $\alpha = (2-d)/(8-d)$, and $\zeta = 3/(8-d)$ in agreement with Hwa and Kardar.[4] More fully, we find $t_l \sim (\nu_\perp^{d-2}/D^2\lambda^4)^{1/(8-d)} \times l^{6/(8-d)}$, $h_l \sim (D^2\lambda^{d-4}/\nu_\perp^{d-2})^{1/(8-d)} l^{(2-d)/(8-d)}$, while $l_\perp \sim (\nu_\perp^3/D\lambda^2)^{1/(8-d)} l^{3/(8-d)}$.

Surface growth with quenched randomness.—So far we have been considering white or colored noise. It is also possible to use this approach to study the influence of quenched random noise on growth far from thermal equilibrium. Fluid flow in porous media is one possible example of surface growth in the presence of quenched noise.[20]

Consider the KPZ equation (1) where the noise $\eta(\mathbf{r},t)$ is replaced by the quenched noise $\eta(\mathbf{r},h)$ with correlation

$$\langle \eta(\mathbf{r},h)\eta(\mathbf{r}',h')\rangle = D''\delta(\mathbf{r}-\mathbf{r}')\delta(h-h').$$

Unlike the original KPZ equation, even when $\lambda = 0$, the equation with quenched noise is nonlinear due to the coupling between h and η and cannot be solved readily.[20] To treat quenched noise we follow the same procedure as for the KPZ equation, except that we assume $\eta_l \sim (D''/l^{d-1}h_l)^{1/2}$. For $\epsilon \ll 1$, i.e., when the term with λ is negligible, we find

$$h_l \sim (D''/\nu^2)^{1/3} l^{(5-d)/3}, \qquad (14)$$

and consequently $\alpha = (5-d)/3$ with $d_c = 5$, and

$$h_t \sim (D''^2/\nu^{d-1})^{1/6} t^{(5-d)/6}, \qquad (15)$$

which implies $\beta = (5-d)/6$ in this regime. Similarly, for $\epsilon \gg 1$, we find $\alpha = (5-d)/5$ with $d_c = 5$, and $\beta = (5-d)/(d+5)$ when $l \gg h_l$, and for rough surfaces we find $\alpha = 4/(d+4)$ and $\beta = 2/(d+2)$ with no upper critical dimension. As far as we know these results are new and have not been studied before either analytically or by simulations.

In conclusion, we have presented a new approach to the study of the scaling behavior of fluctuations in dissipative dynamical systems. We have illustrated the method by applying it to a number of systems that have been investigated recently as well as to the problem of surface growth with quenched noise. In particular, we have shown how the expressions conjectured by Kim and Kosterlitz for the scaling exponents in the KPZ equation may be derived using this approach. In addition to providing a method for determining the scaling exponents of complex nonlinear equations, this approach provides insight into the scaling regimes that can be observed in the microscopic parameter space of different systems. The different scaling regimes manifest themselves in regions where a particular term in the equation becomes relevant. The type of arguments used in this approach are quite similar to those used in Flory theory for equilibrium systems. Therefore, due to the lack of standard methods for studying nonequilibrium phenomena, this approach will be useful in the study of a wide variety of related problems.

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