Semirigid Supergravity

Jacques Distler
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544

Philip Nelson
Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 22 January 1991)

We formulate two-dimensional topological gravity starting from *local* $N=2$ superconformal geometry. The theory is free from the very beginning. The usual "twisting" of the $N=2$ algebra emerges from symmetry breaking when we expand about a nonzero value for one of the ghost fields. The mysterious linear term in the supercurrent emerges automatically, as does a full superfield formalism for the whole system including ghosts. We analyze the moduli space of the "semirigid" super Riemann surfaces associated with this theory, including their allowed degenerations.

PACS numbers: 11.30.Pb, 04.65.+e

Two-dimensional topological gravity still has many mysterious features. On one hand, it is supposed to be a cohomological quantum field theory, like topological Yang-Mills (TYM) theory.[1] From this viewpoint it makes sense to construct it from a double complex with differentials Q_S and Q_V corresponding respectively to a scalar supersymmetry and the usual Becchi-Rouet-Stora-Tyutin (BRST) charge of the symmetry algebra of conformal gravity, in this case the Virasoro algebra. This approach was followed in Refs. 2–8 and elsewhere. A related, but distinct, construction obtains the theory from Chern-Simons theory for the superanalog of $SL(2,\mathbb{R})$.[9] It seems remarkable that after a rather arduous gauge-fixing procedure only a very small set of fields remain in the end, and these are *free*.

On the other hand, it is by now quite clear that topological gravity has an intimate association with $N=2$ superconformal symmetry. The matter systems which can couple to pure topological gravity invariably seem to come from "twisting" systems with $N=2$ supersymmetry (SUSY).[10–14] If this is not an accident then we should see this symmetry entering in some essential way in the geometry underlying this system, and, in particular, in the gravity sector. Indeed there are hints, but they raise as many questions as they answer.

Verlinde and Verlinde have shown that the same "twisted" $N=2$ symmetry algebra just mentioned also acts on the ghost and Liouville systems of topological gravity, which are all that remain of the gravity sector after gauge fixing.[9] But what physically is responsible for this twisting? If only a fragment of local $N=2$ SUSY remains, what has broken it? Is there another phase of the theory where this symmetry is not broken? The question is particularly acute because the twisted algebra controls even the pure gravity theory; we cannot appeal to the matter system for help in breaking the $N=2$ symmetry. We do not know of another system where gravity breaks its own supersymmetries, leaving *unbroken* the ordinary coordinate group (or its conformal subgroup).

A second question relates to the specific form of the residual symmetry generators L_n, G_n, and Q_S found in Ref. 9. While L_n and Q_S are *bilinear* in fields (like any Noether charge in free field theory), one sees that to make the theory work one needs a *linear* term in G_n. In Refs. 9 and 15 this term is added by hand, but it should come out of some symmetry principle.[16]

Third, in Refs. 14 and 15 an elegant superspace construction was proposed in which matter fields and allowed vertex operators assemble into superconformal fields. Unfortunately the ghost fields b, β, c, γ do not seem to form superfields, and hence the generators L_n, G_n, Q_S do not have superfield forms. This seriously limits the utility of superspace; we would like to use super-contour-deformation arguments and so on. Moreover, it is hard to see how the ghosts, with the inhomogeneous transformation law implied above, can be tensors on *any* super Riemann surface. These puzzles cast doubt on whether the supermoduli space corresponding to the super Riemann surfaces of Refs. 14 and 15 is the right one to integrate over. The statement that this space was trivially fibered over ordinary moduli space played a role in the analyses of Refs. 8 and 15, and so we need to be sure it is right. In particular, it is crucial to understand the degeneration of these surfaces.

In this Letter we will address all the above issues. Our point of departure is extremely simple. We begin with *local* $N=2$ superconformal gravity.[17] The corresponding geometry was studied by Cohn.[18] We then find a field in the gravity sector (there is no matter) whose vacuum expectation value breaks the full symmetry to the twisted subalgebra. There is only one field available which can do this job without spoiling ordinary conformal invariance. We describe the corresponding reduced geometry, which we have dubbed "semirigid" geometry, because half of the local $N=2$ supersymmetries get broken down to rigid SUSY. This may seem strange, since in $N=1$ there is no such thing as rigid SUSY on an arbitrary Riemann surface. In $N=2$, however, we can consistently attribute spin zero to one supercharge, and hence give invariant meaning to its zero mode.

Remarkably, many of the mysterious elements of the complicated cohomological-field-theory construction emerge automatically in this approach. For example, both ghost and vertex-operator superfields can coexist in this geometry.[19] And due to the vacuum expectation value, all the ghost parts of L_n, G_n, and Q_S including the linear term come directly from the unbroken bits of the full $N=2$ stress tensor. We will *not* inquire into the dynamical origin of the symmetry breaking proposed here, however. This question is, of course, the great mystery of any topological gravity theory. Finally, we describe some of the main features of the supermoduli spaces $\hat{\mathcal{M}}$ appropriate to semirigid geometry. We will prove that $\hat{\mathcal{M}}$ is indeed split (i.e., trivially fibered) over the ordinary $\mathcal{M}$.

Our simple construction thus puts $N=2$ supersymmetry at center stage. The cohomological interpretation is a by-product due to the fact that Q_S+Q_V happens to be an invariant, nilpotent operator, where again Q_V is the Virasoro BRST operator. In contrast, TYM theory has a superspace version where $Q_S=\partial/\partial\theta$ is gauge invariant all by itself.[6,20] The difference is as usual that gravity is not quite a gauge theory; its symmetry generators get tangled up in spacetime. This difficulty seems responsible for the complexity of the cohomological approach to topological gravity.

To get started we recall some $N=2$ geometry, beginning with a coordinate-invariant definition of $N=2$ super Riemann surfaces (SRS).[21] An $N=2$ super Riemann surface $\hat{\Sigma}$ is patched from pieces of $\mathbb{C}^{1|2}$. Generalizing the $N=1$ discussion of Ref. 22 we take $\hat{\Sigma}$ to be equipped with two odd line bundles $\mathcal{D},\tilde{\mathcal{D}}\subseteq T\hat{\Sigma}$, and require $\mathcal{D},\tilde{\mathcal{D}}$ each to be integrable: $[D,D']\propto D$ for any sections D,D', of $\mathcal{D}$, etc. We also require that $[D,\tilde{D}]$ be linearly independent of D and $\tilde{D}$. Then there are local coordinates $\mathbf{z}\equiv(z,\theta,\xi)$ such that $D_\theta\equiv\partial_\theta+\xi\partial_z$ and $\tilde{D}_\xi=\partial_\xi+\theta\partial_z$ span $\mathcal{D}$ and $\tilde{\mathcal{D}}$, respectively. Any other $\mathbf{z}'=(z',\theta',\xi')$ are related by a transformation preserving D and $\tilde{D}$ up to a multiplier:[18]

$$\begin{aligned}z'&=f+\theta a\rho+\xi e\alpha+\theta\xi\,\partial(\alpha\rho)\,,\\ \theta'&=\alpha+\theta a+\theta\xi\,\partial\alpha\,,\\ \xi'&=\rho+\xi e-\theta\xi\,\partial\rho\,,\end{aligned}\tag{1}$$

where f,a,e are commuting functions of z, α,ρ are anticommuting functions of z, $\partial=\partial/\partial z$, and we impose

$$ea=\partial f+\alpha\,\partial\rho+\rho\,\partial\alpha\,.$$

Thus there are two independent even and two odd sets of symmetry generators, the usual $L_n,J_n,G_n,\tilde{G}_n$.

Under these superconformal transformations we get $D_\theta=(D_\theta\theta')D_{\theta'}$ and $\tilde{D}_\xi=(D_\xi\xi')D_{\xi'}$, which define the transition functions of the bundles $\mathcal{D},\tilde{\mathcal{D}}$. Henceforth a section of $\mathcal{D}^{\otimes n}$ will be denoted by a component function with n raised θ indices, etc. A $\theta\xi$ index pair will be rewritten as a z index. Infinitesimal superconformal transformations are generated by

$$V_v\equiv v\partial_z+\tfrac{1}{2}(Dv)\tilde{D}+\tfrac{1}{2}(\tilde{D}v)D\,,$$

where $v\equiv v^z(\mathbf{z})$ is an even tensor field. We find $[V_{v_1},V_{v_2}]=V_{[v_1,v_2]}$, where

$$[v_1,v_2]=v_1\partial v_2-v_2\partial v_1+\tfrac{1}{2}Dv_1\tilde{D}v_2+\tfrac{1}{2}\tilde{D}v_1Dv_2\,.\tag{2}$$

Finally, computing the Berezin determinant of (1) shows that the volume form $d\mathbf{z}\equiv[dz|d\theta d\xi]$ is *invariant.* Thus the integral $\oint d\mathbf{z}$ sets up a Serre duality between (p,q) tensors and $(-p,-q)$ tensors, unlike $N=0$ or 1.

The C ghost is always determined by the group of allowed coordinate transformations, so we have $C\equiv C^z(\mathbf{z})$. The stress tensor and B ghost are always dual to v, so $B\equiv B_z(\mathbf{z})$, $T\equiv T_z(\mathbf{z})$. We will choose to expand these fields as

$$\begin{aligned}C^z&=c^z+\theta\gamma^\xi+\xi\tilde{\gamma}^\theta+\theta\xi\tilde{c}\,,\\ B_z&=\tilde{b}_z+\theta\tilde{\beta}_{\theta z}-\xi\beta_{\xi z}-\theta\xi(b_{zz}+\partial_z\tilde{b}_z)\,,\\ T_z&=J_z+\theta\tilde{G}_{\theta z}-\xi G_{\xi z}+\theta\xi[(T_B)_{zz}+\partial_zJ_z]\,.\end{aligned}\tag{3}$$

These peculiar linear combinations will be useful momentarily. The operator products of the component fields yield

$$B(\mathbf{z}_1)C(\mathbf{z}_2)=\theta_{12}\xi_{12}/z_{12}=C(\mathbf{z}_1)B(\mathbf{z}_2)\,,$$

where

$$z_{12}\equiv z_1-z_2-\theta_1\xi_2-\xi_1\theta_2\,,$$

$$\theta_{12}\equiv\theta_1-\theta_2,\quad \xi_{12}\equiv\xi_1-\xi_2$$

are elementary translation-invariant functions.[18] The ghost stress tensor can be constructed by the method of Ref. 23: demanding that $[T[v],\phi]=\mathcal{L}_v\phi$ for $\phi=B$, C, and $\mathcal{L}$ the Lie bracket implied by (2) gives

$$T=\partial(CB)-\tfrac{1}{2}(DB\tilde{D}C+\tilde{D}BDC)\,.\tag{4}$$

We now explore how to break this large symmetry. Suppose that instead of asking $D_{\theta'}\propto D_\theta$ we require $D_{\theta'}=D_\theta$. In other words, we require that $\mathcal{D}$ be *trivial,* and a global section D be given. (Note that this precludes twisting in the sense of Ref. 18.) In $N=1$ such a constraint would break us all the way down to rigid Poincaré SUSY. Now, however, we find merely that in (1) we need $\alpha\equiv$const, $a\equiv1$. Infinitesimally this says that $\tilde{D}v\equiv$const. Examining (1) we see that it now makes sense to attribute spin 0 to θ and spin 1 to ξ. Thus in (3) the only Bose field whose vacuum expectation value can break $N=2$ in this way is $\tilde{\gamma}^\theta$. More invariantly we break

$N=2$ by the constraint

$$\tilde{D}C \equiv q \quad \text{or} \quad \check{\gamma}=q, \quad \check{c}=\partial c. \tag{5}$$

Since $\tilde{D}_\xi C^{\theta\xi}$ is a section of $\mathcal{D}$ we again supplied a trivialization. q is some constant.

One readily finds the unbroken generators to be the modes L_n, G_n, $\tilde{G}_0$ of T_B, G, $\tilde{G}$ in (3), and these obey the required "twisted $N=2$" algebra by virtue of (2). Substituting (5) into (4) we find

$$T_B = -2b\,\partial c - (\partial b)c - 2\beta\,\partial\gamma - (\partial\beta)\gamma\,,$$

$$G = -2\beta\,\partial c - (\partial\beta)c + \tfrac{1}{2}qb\,,$$

$$Q_S \equiv 2\tilde{G}_0 = \oint \gamma b\,.$$

Choosing $q=-2$ yields the formulas of Ref. 15 after some trivial changes of notation. Note that we did not *constrain* the B field at all. The unfamiliar new components $\check{b}$, $\check{\beta}$ simply dropped out of the expressions for the unbroken charges; they remain in the other charges, which, however, are not symmetries. Nevertheless, we cannot represent the BC system by ordinary superfields on any $1|1$ superspace.

Because Q_S has nontrivial commutator with G_0 we cannot regard it as a global charge, any more than L_0 is global on an ordinary Riemann surface.[24] However, the combination $Q_T = Q_S + Q_V$ *does* commute with the unbroken charges, and so defines the global nilpotent charge needed to get a topological field theory.[1]

Let us turn to the moduli of semirigid super Riemann surfaces, or SSRS. We find the moduli by examining the Čech cohomology of the allowed coordinate transformations (see, e.g., Ref. 25). Since we want the dimension of moduli space to be $3g-3|3g-3$, we further restrict (1) to $\alpha=0$, $a=1$:

$$\begin{aligned} z' &= f(z) + \theta\rho(z)\,, \\ \theta' &= \theta\,, \\ \xi' &= \rho(z) + \xi\,\partial f(z) - \theta\xi\,\partial\rho(z)\,. \end{aligned} \tag{6}$$

Infinitesimally this means $\tilde{D}v=0$ or $v^z = v_0^{\bar{z}} + \theta v^z + \theta\xi \times \partial_z v_0^{\bar{z}}$ and the desired dimension follows. We next note that since the bundle $\tilde{\mathcal{D}} \subseteq T\hat{\Sigma}$ is integrable, we can reduce to a manifold with coordinates $z|\theta$ by modding out the flow of $\tilde{D}$. The transition functions then reduce to

$$z' = f(z) + \theta\rho(z), \quad \theta' = \theta\,. \tag{7}$$

Conversely, given patching functions of the form (7) we uniquely get a SSRS. Then, e.g., any scalar function $f = f + \theta\phi$ lifts uniquely to the chiral superfield $F(\mathbf{z}) = f + 2\theta\phi + \theta\xi\,\partial f$. Similarly we may instead mod out by the flow of D, reducing (6) to the form given in Ref. 14:

$$z' = f(z)\,, \quad \xi' = \partial f(z) + \rho(z)\,. \tag{8}$$

Again any set of transition functions of this form induce a unique SSRS by substituting f,ρ into (6). The relation between (6), (7), and (8) is preserved under composition of maps. Thus (6), (7), and (8) all define exactly the *same* moduli space.

Writing any of (6),(7),(8) as $\mathbf{z}_i = F_{ij}(\mathbf{z}_j)$, we see that the condition $F_{ki} = F_{kj} \circ F_{ji}$ implies

$$f_{ki} = f_{kj} \circ f_{ji}\,,$$

$$\rho_{ki} = (\partial f_{kj} \circ f_{ji}) \cdot \rho_{ji} + \rho_{kj} \circ f_{ji}\,.$$

These say that $\{f_{ij}\}$ define an ordinary Riemann surface while $\{\rho_{ij}\}$ define a cocycle in its tangent space. Hence it makes sense to choose f_{ij} to depend only on the commuting moduli, and ρ_{ki} to be linear in the anticommuting moduli, in contrast to the situation in $N=1$. This in turn implies that the moduli space $\hat{\mathcal{M}}$ is split over $\mathcal{M}$, at least away from the boundary; details will appear elsewhere.

The correct notion of puncture on any sort of SRS is[25] that of a divisor of codimension $1|0$ obtained from z by some allowed coordinate transformation of the disk. In the language of (7) we find divisors of the form $z - z_0 - \theta\theta_0$, yielding $1|1$ new moduli for a total of $3g-3+N|3g-3+N$ if there are N punctures. Counting then shows that the correct plumbing fixture for degeneration of SSRS must have $1|1$ pinching parameters, similarly to spin pinches in $N=1$ (but unlike super pinches).[26] The correct choice turns out to be to join the $z|\theta$ plane to the $u|\zeta$ plane via

$$z = u^{-1}(q + \zeta\delta)\,, \quad \theta = \zeta\,. \tag{9}$$

For these assignments to make sense we need to know that the three-punctured sphere is rigid.[27] This follows from the observation of a $3|3$ parameter group of sphere automorphisms:

$$z' = \frac{(a+\theta\alpha)z + (b+\theta\beta)}{(c+\theta\gamma)z + (d+\theta\delta)}\,, \quad \theta' = \theta\,,$$

where we may take $d=1$, $\delta=0$. Note that (9) is of this form.

Now that we know the moduli space one can elaborate the full machinery of constructing the superoperator formalism and the string measure for the insertion of weakly physical states.[28,29] Alternately one may try to covariantize the vertex operators by inventing a Liouville sector as in Ref. 9.

We have provided a concrete framework for $d=2$ topological gravity by starting with *local* $N=2$ superconformal gravity. This system is not conformally invariant by itself, but it becomes so once we drop the modes $\check{c}$, $\check{\gamma}$ fixed by a symmetry-breaking constraint. Thus the broken theory requires no Liouville sector. Indeed it is born free, rather than appearing interacting at first and mysteriously becoming free after gauge fixing. The system has a global nilpotent charge and hence is topological. It explains why the appropriate matter systems always seem to have twisted $N=2$ SUSY, and other mysteries as well. It also urges us to ask about the bigger mystery of the dynamical origin of such a symmetry breaking.

We would like to thank J. Cohn, R. Dijkgraaf, S. Giddings, H. S. La, A. Morozov, C. Vafa, E. Witten, E. Wong, and especially E. Verlinde and H. Verlinde for valuable discussions. This work was supported in part by NSF Grants No. PHY88-57200 and No. PHY80-19754 and by the A. P. Sloan Foundation.

Note added.—Recently, we have found that the passage from Eq. (6) to (8) is a special case of a construction given by Dolgikh, Rosly, and Schwarz.[30] The geometrical framework given in this paper can be used to derive recursion formulas such as the dilaton equation, again without recourse to any Liouville sector.[31]

[1]E. Witten, Commun. Math. Phys. **117**, 353 (1988).

[2]E. Witten, Nucl. Phys. **B342**, 281 (1989).

[3]J. Labastida, M. Pernici, and E. Witten, Nucl. Phys. **B310**, 611 (1988).

[4]D. Montano and J. Sonnenschein, Nucl. Phys. **B313**, 258 (1989); **B324**, 348 (1989).

[5]R. Myers and V. Periwal, Nucl. Phys. **B333**, 536 (1990).

[6]S. Ouvry, R. Stora, and P. van Baal, Phys. Lett. B **220**, 159 (1989).

[7]L. Baulieu and I. Singer, Commun. Math. Phys. **135**, 253 (1991).

[8]S. Giddings, University of California, Santa Barbara, Report No. UCSBTH-90-63, 1990 (to be published).

[9]E. Verlinde and H. Verlinde, Nucl. Phys. **B348**, 457 (1991).

[10]E. Witten, Commun. Math. Phys. **118**, 411 (1988).

[11]T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A **5**, 1693 (1990).

[12]K. Li, Caltech Reports No. CALT-68-1662 and No. CALT-68-1670, 1990 (to be published).

[13]C. Vafa, Harvard University Report No. HUTP-90-A064, 1990 (to be published).

[14]R. Dijkgraaf, H. Verlinde, and E. Verlinde, Nucl. Phys. B (to be published).

[15]R. Dijkgraaf, H. Verlinde, and E. Verlinde, Princeton University Report No. PUPT-1217, 1990 (to be published).

[16]A different suggestion can be found in Ref. 8.

[17]M. Ademollo *et al.*, Phys. Lett. **62B**, 105 (1976); Nucl. Phys. **B111**, 77 (1976).

[18]J. Cohn, Nucl. Phys. **B284**, 349 (1987).

[19]We are grateful to H. Verlinde for suggesting this.

[20]J. Horne, Nucl. Phys. **B318**, 22 (1989).

[21]For reasons to become clear we will not allow the "twisting" considered in Ref. 18.

[22]Yu. Manin, Funk. Anal. **20**, 88 (1986) [Funct. Anal. Appl. **20**, 244 (1987)].

[23]D. Friedan, in *Unified String Theories,* edited by M. Green and D. Gross (World Scientific, Singapore, 1986).

[24]Q_S is scalar under ordinary L_n transformations, but this is not good enough.

[25]P. Nelson, in *Particles, Strings, and Supernovae,* edited by A. Jevicki and C.-I. Tan (World Scientific, Singapore, 1989).

[26]J. Cohn, Nucl. Phys. **B306**, 239 (1988).

[27]This is analogous to the sphere in $N=1$ geometry with two spin punctures and one superpuncture (but unlike the case with three superpunctures).

[28]P. Nelson, Phys. Rev. Lett. **62**, 993 (1989).

[29]J. Distler and P. Nelson, Commun. Math. Phys. (to be published).

[30]S. Dolgikh, A. Rosly, and A. Schwarz, Commun. Math. Phys. **135**, 91 (1990).

[31]J. Distler and P. Nelson, Princeton University Report No. PUPT-1232=UPR-0428T, 1990 (to be published).