Coupled Alfvén Waves in a Quadrupole Magnetic Field and Mode Conversion

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Coupled Alfvén waves propagating along a quadrupole magnetic field are studied. It is shown analytically that slow and fast Alfvén waves mode couple due to the spatial modulation of the quadrupole magnetic field, and one Alfvén mode can convert to another one. The process is analogous to a resonant parametric mode coupling. The coupling coefficient depends on the ellipticity of the flux surface and vanishes when the magnetic field is axisymmetric.

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It is of importance to study wave propagation in order to understand the effects of waves on rf heating, transport, and stability in magnetically confined plasmas. Especially, the propagation in nonuniform magnetic fields such as those that occur in mirrors and helical systems is very interesting as it is associated with complex problems in wave reflection and mode conversion.

In this Letter we study axial propagation of Alfvén waves in a quadrupole magnetic field analytically. This study is motivated by an attempt to understand the experimental results on the mode conversion of Alfvén waves in GAMMA 10.[1] We show here that two Alfvén modes, i.e., fast and slow waves, mode couple due to the spatial modulation of a quadrupole magnetic field, and one Alfvén mode can convert to the other. The coupled-mode equations between fast and slow waves via the spatial quadrupole field modulation are derived. The coupling coefficient depends on the ellipticity of the flux surface, and then vanishes when the magnetic field is axisymmetric. If we suppose the spatial quadrupole field modulation to be a "virtual" mode with zero frequency, we can see that the coupling process is analogous to a resonant parametric mode coupling among the two Alfvén modes and a virtual mode and that efficient mode conversion takes place when a resonant condition with respect to axial wave number among these modes is satisfied. This is a new mechanism for mode conversion of Alfvén waves.

The starting point is the Maxwell wave equation given by

$$\nabla\times\nabla\times\mathbf{E}-(\omega/c)^2\overleftrightarrow{\varepsilon}\mathbf{E}=0\,, \qquad (1)$$

where $\overleftrightarrow{\varepsilon}$ is the plasma dielectric tensor, which is calculated from a cold plasma model, ω is the wave frequency, and c is the light speed. When we use flux coordinates (ψ,θ,z), the magnetic field is expressed as $\mathbf{B}=B\mathbf{b}=\nabla\psi\times\nabla\theta$, and the wave electric field as $\mathbf{E}=E_\psi\nabla\psi+E_\theta\nabla\theta+E_z\mathbf{b}$ in the covariant form. Then we obtain, from Eq. (1),

$$\frac{\partial}{\partial z}\left[\frac{|\nabla\psi|^2}{B}\frac{\partial E_\psi}{\partial z}+\frac{(\nabla\psi\cdot\nabla\theta)}{B}\frac{\partial E_\theta}{\partial z}\right]-B\frac{\partial}{\partial\theta}\left[\frac{\partial E_\theta}{\partial\psi}-\frac{\partial E_\psi}{\partial\theta}\right]+\left(\frac{\omega}{c}\right)^2(S-1)\frac{|\nabla\psi|^2}{B}E_\psi+\left(\frac{\omega}{c}\right)^2\left[(S-1)\frac{(\nabla\psi\cdot\nabla\theta)}{B}-iD\right]E_\theta=0\,, \qquad (2)$$

$$\frac{\partial}{\partial z}\left[\frac{|\nabla\theta|^2}{B}\frac{\partial E_\theta}{\partial z}+\frac{(\nabla\psi\cdot\nabla\theta)}{B}\frac{\partial E_\psi}{\partial z}\right]+B\frac{\partial}{\partial\psi}\left[\frac{\partial E_\theta}{\partial\psi}-\frac{\partial E_\psi}{\partial\theta}\right]+\left(\frac{\omega}{c}\right)^2(S-1)\frac{|\nabla\theta|^2}{B}E_\theta+\left(\frac{\omega}{c}\right)^2\left[(S-1)\frac{(\nabla\psi\cdot\nabla\theta)}{B}+iD\right]E_\psi=0\,, \qquad (3)$$

with

$$S=1-\sum_{e,i}\frac{\omega_p^2}{\omega^2-\omega_c^2}\approx1+\frac{c^2}{V_A^2}\frac{1}{1-(\omega/\omega_{ci})^2}\,, \qquad (4)$$

$$D=\sum_{e,i}\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2-\omega_c^2}\approx-\frac{c^2}{V_A^2}\frac{\omega_{ci}/\omega}{1-(\omega/\omega_{ci})^2}\,, \qquad (5)$$

where ω_c is the cyclotron frequency, ω_p the plasma frequency, and V_A the Alfvén velocity. Here E_z is neglected as $E_z \ll E_\psi$ and E_θ when $\omega\sim\omega_{ci}$. $|\nabla\psi|^2$, $|\nabla\theta|^2$, and $(\nabla\psi\cdot\nabla\theta)$ appear in Eqs. (2) and (3) since the covariant basis vectors $\nabla\psi$ and $\nabla\theta$ are neither unitary nor orthogonal.

We express the magnetic field line of a quadrupole field by $x(z)=\sigma(z)x_0$ and $y(z)=\tau(z)y_0$. Here (x_0,y_0) denotes the radial position at $z=z_0$ where the flux surface is circular. The x_0 and y_0 are expressed as x_0

$=(2\psi/B_0)^{1/2}\cos\theta$ and $y_0=(2\psi/B_0)^{1/2}\sin\theta$, respectively, and satisfy $(x/\sigma)^2+(y/\tau)^2=x_0^2+y_0^2=r_0^2=2\psi/B_0$, where $B_0=B(z_0)$. Then $|\nabla\psi|^2$, $|\nabla\theta|^2$, and $(\nabla\psi\cdot\nabla\theta)$ are expressed as

$$\begin{aligned}|\nabla\psi|^2&=(2\psi B_0)[h(z)+\alpha(z)\cos 2\theta]\,,\\ |\nabla\theta|^2&=(B_0/2\psi)[h(z)-\alpha(z)\cos 2\theta]\,,\\ (\nabla\psi\cdot\nabla\theta)&=-B_0\alpha(z)\sin 2\theta\,,\end{aligned} \tag{6}$$

with $h(z)=\frac{1}{2}(1/\sigma^2+1/\tau^2)$ and $\alpha(z)=\frac{1}{2}(1/\sigma^2-1/\tau^2)$.

We find that mode coupling with respect to the azimuthal mode number arises due to the θ dependence of $|\nabla\psi|^2$, $|\nabla\theta|^2$, and $(\nabla\psi\cdot\nabla\theta)$ in Eqs. (2) and (3). One mode with the mode number m couples to excite other modes with $m\pm 2$ through the quadrupole magnetic-field component with the mode number of ± 2. If we Fourier expand E_ψ and E_θ as

$$\begin{aligned}E_\psi(\psi,\theta,z)&=\sum_{m=-\infty}^{\infty}F_m(\psi,z)\exp(im\theta)\,,\\ E_\theta(\psi,\theta,z)&=\sum_{m=-\infty}^{\infty}G_m(\psi,z)\exp(im\theta)\,,\end{aligned} \tag{7}$$

we obtain coupled-mode equations for Fourier components F_m and G_m as follows:

$$\begin{aligned}&\frac{\partial}{\partial z}\left[2\psi p\frac{\partial}{\partial z}F_m\right]-Bm\left[mF_m+i\frac{\partial}{\partial\psi}G_m\right]+\left[\frac{\omega}{c}\right]^2[(S-1)2\psi pF_m-iDG_m]+\frac{\partial}{\partial z}\left[2\psi\beta\frac{\partial}{\partial z}F_{m-2}+i\beta\frac{\partial}{\partial z}G_{m-2}\right]\\ &\quad+\left[\frac{\omega}{c}\right]^2(S-1)\beta[2\psi F_{m-2}+iG_{m-2}]+\frac{\partial}{\partial z}\left[2\psi\beta\frac{\partial}{\partial z}F_{m+2}-i\beta\frac{\partial}{\partial z}G_{m+2}\right]\\ &\quad+\left[\frac{\omega}{c}\right]^2(S-1)\beta[2\psi F_{m+2}-iG_{m+2}]=0\,,\end{aligned} \tag{8}$$

$$\begin{aligned}&\frac{\partial}{\partial z}\left[\frac{1}{2\psi}p\frac{\partial}{\partial z}G_m\right]+B\frac{\partial}{\partial\psi}\left[\frac{\partial}{\partial\psi}G_m-imF_m\right]+\left[\frac{\omega}{c}\right]^2\left[(S-1)\frac{p}{2\psi}G_m+iDF_m\right]-\frac{\partial}{\partial z}\left[\frac{1}{2\psi}\beta\frac{\partial}{\partial z}G_{m-2}-i\beta\frac{\partial}{\partial z}F_{m-2}\right]\\ &\quad-\left[\frac{\omega}{c}\right]^2(S-1)\beta\left[\frac{1}{2\psi}G_{m-2}-iF_{m-2}\right]-\frac{\partial}{\partial z}\left[\frac{1}{2\psi}\beta\frac{\partial}{\partial z}G_{m+2}+i\beta\frac{\partial}{\partial z}F_{m+2}\right]\\ &\quad-\left[\frac{\omega}{c}\right]^2(S-1)\beta\left[\frac{1}{2\psi}G_{m+2}+iF_{m+2}\right]=0\,,\end{aligned} \tag{9}$$

where

$$\begin{aligned}p(z)&=h(z)[B_0/B(z)]=\tfrac{1}{2}(\Theta+1/\Theta)\,,\\ 2\beta(z)&=\alpha(z)[B_0/B(z)]=\tfrac{1}{2}(\Theta-1/\Theta)\,,\end{aligned}$$

and $\Theta=\tau/\sigma$.

We now restrict ourselves to the Alfvén-wave propagation along the quadrupole magnetic field and treat the limit where $\nabla\approx\mathbf{z}\partial/\partial z$, for simplicity. If we define X_m and Y_m by

$$X_m=G_m-2i\psi F_m,\quad Y_m=-G_m-2i\psi F_m\,, \tag{10}$$

X_m and Y_m denote left-hand and right-hand circularly polarized wave components, respectively. Then Eqs. (8) and (9) are rewritten as follows:

$$\frac{d}{dz}\left[p\frac{d}{dz}X_m\right]+k_s^2X_m+\frac{d}{dz}\left[2\beta\frac{d}{dz}Y_{m+2}\right]+2\beta k_T^2Y_{m+2}=0\,, \tag{11}$$

$$\frac{d}{dz}\left[p\frac{d}{dz}Y_m\right]+k_F^2Y_m+\frac{d}{dz}\left[2\beta\frac{d}{dz}X_{m-2}\right]+2\beta k_T^2X_{m-2}=0\,, \tag{12}$$

with

$$\begin{aligned}k_s^2&=(\omega/V_A)^2(p+\omega/\omega_{ci})/[1-(\omega/\omega_{ci})^2]\,,\\ k_F^2&=(\omega/V_A)^2(p-\omega/\omega_{ci})/[1-(\omega/\omega_{ci})^2]\,,\\ k_T^2&=(\omega/V_A)^2/[1-(\omega/\omega_{ci})^2]\,.\end{aligned} \tag{13}$$

Equations (11) and (12) describe the mode coupling between X_m and Y_{m+2} (or between X_{m-2} and Y_m) due to the spatial modulation of the quadrupole magnetic field. If the magnetic field is axisymmetric, the coupling coefficient β vanishes as $\Theta=1$ ($\sigma=\tau$), and then X_m and Y_{m+2} decouple with each other.

In order to solve Eqs. (11) and (12) analytically, we consider the Alfvén-wave propagation in a weak periodic quadrupole-field mirror described by $\sigma(z)=1-\varepsilon\times\cos(k_0z)$ and $\tau(z)=1+\varepsilon\cos(k_0z)$ with $\varepsilon\ll 1$. Here ε and k_0 are the amplitude and axial wave number of the quadrupole-field modulation, respectively. For $\varepsilon\ll 1$, $B_0/B(z)\approx 1$, $h(z)\approx 1$, and $\alpha(z)\approx 2\varepsilon\cos(k_0z)$. Then we have $p(z)\approx 1$ and $\beta(z)\approx\varepsilon\cos(k_0z)$. Using these relations, we obtain the following coupled equations between the $m=-1$ slow wave X_{-1} ($\equiv X$) with left-hand polarization and the $m=+1$ fast wave Y_{+1} ($\equiv Y$) with right-hand polarization as follows:

$$\left[\frac{d^2}{dz^2}+k_s^2\right]X+2\varepsilon\left[\frac{d}{dz}\left[\cos(k_0z)\frac{dY}{dz}\right]+k_T^2\cos(k_0z)Y\right]=0\,, \qquad (14)$$

$$\left[\frac{d^2}{dz^2}+k_F^2\right]Y+2\varepsilon\left[\frac{d}{dz}\left[\cos(k_0z)\frac{dX}{dz}\right]+k_T^2\cos(k_0z)X\right]=0\,. \qquad (15)$$

These equations are analogous to parametric coupled-mode equations,[2] when we suppose the spatial modulation of the quadrupole magnetic field, i.e., $2\varepsilon\cos(k_0z)$, to be a virtual wave with zero frequency and the axial wave number k_0.

Fourier transforming Eqs. (14) and (15), we obtain

$$D_s(k)X(k)=\varepsilon[k(k+k_0)-k_T^2]Y(k+k_0)+\varepsilon[k(k-k_0)-k_T^2]Y(k-k_0)\,, \qquad (16)$$

$$D_F(k)Y(k)=\varepsilon[k(k+k_0)-k_T^2]X(k+k_0)+\varepsilon[k(k-k_0)-k_T^2]X(k-k_0)\,, \qquad (17)$$

with $D_s(k)=k_s^2-k^2$ and $D_F(k)=k_F^2-k^2$. Here $X(k\pm k_0)$ can be approximated as follows:

$$D_s(k+k_0)X(k+k_0)=\varepsilon[k(k+k_0)-k_T^2]Y(k)\,, \qquad (18)$$

$$D_s(k-k_0)X(k-k_0)=\varepsilon[k(k-k_0)-k_T^2]Y(k)\,, \qquad (19)$$

since $Y(k\pm 2k_0)$ is of the higher order to be neglected. Substituting Eqs. (18) and (19) into Eq. (17), we obtain the following parametric dispersion equation:

$$D_F(k)=\varepsilon^2\left[\frac{[k(k+k_0)-k_T^2]^2}{D_s(k+k_0)}+\frac{[k(k-k_0)-k_T^2]^2}{D_s(k-k_0)}\right]. \qquad (20)$$

We can solve Eq. (20) under the assumption of the resonant condition $k_0=k_s-k_F$. For $k\simeq k_F$, $D_F(k)\simeq 0$, $D_s(k+k_0)\simeq 0$, and $D_s(k-k_0)$ is off resonant. Then neglecting $D_s(k-k_0)$, we obtain the eigenvalue of k as follows:

$$k=k_F\pm\Delta,\quad \Delta=\varepsilon(k_T^2-k_sk_F)/2(k_sk_F)^{1/2}. \qquad (21)$$

Propagating coupled solutions are approximately given by

$$Y=Y_0\cos(\Delta z)\exp(ik_Fz)\,, \qquad (22)$$

$$X=i(k_F/k_s)^{1/2}Y_0\sin(\Delta z)\exp(ik_sz)\,, \qquad (23)$$

where Y_0 is the wave amplitude of the fast wave in the absence of the slow wave. Equations (22) and (23) show that the slow- (fast-) wave amplitude increases with the decrease of the fast- (slow-) wave amplitude. Then we can see that these solutions express the mode conversion between the slow and fast waves and satisfy a conservation relation given by

$$k_F|Y_0|^2=k_F|Y|^2+k_s|X|^2\,. \qquad (24)$$

The relation (24) is analogous to the Manley-Rowe relation. It can be also shown that the efficiency of the mode conversion is significantly reduced if the axial wave number k is far from the resonant condition.

We now compare the present analytical results on the mode conversion of Alfvén waves with the experimental results on the polarization reversal of Alfvén waves observed in GAMMA 10.[1] In the experiments, the $m=+1$ fast wave is excited in the axisymmetric central cell and observed to be right-hand polarized in a core region. The wave field observed in the anchor cell with a quadrupole magnetic field is left-hand polarized in the core region and strong ion heating in the anchor cell is also observed, which is supposed to be due to the fundamental ion-cyclotron-resonance heating. Therefore, the wave observed in the anchor cell is considered to be the $m=-1$ slow wave. On the other hand, when the $m=-1$ slow wave is excited in the central cell, the wave field with right-hand polarization and no ion heating are observed in the anchor cell. From these results, the existence of an efficient mode conversion is strongly suggested. The present analyses show that the $m=+1$ fast ($m=-1$ slow) wave can mode convert to the $m=-1$ slow ($m=+1$ fast) wave due to the spatial modulation of the quadrupole magnetic field, in agreement with the experimental observations. Then we can conclude that the observed polarization reversal of Alfvén waves results from the mode conversion from the $m=+1$ fast ($m=-1$ slow) wave to the $m=-1$ slow ($m=+1$ fast) wave.

Recently, several experimental works on the mode conversion of Alfvén waves have been reported.[3-5] These experiments have been done in axisymmetric mirror devices. However, the present mode conversion is induced by the quadrupole magnetic field and then does not occur when the magnetic field is axisymmetric. Therefore, we note that this is a new mechanism for the mode conversion of Alfvén waves.

In summary, we studied coupled Alfvén waves propagating in a quadrupole magnetic field analytically. We showed that slow and fast Alfvén waves mode couple due to the spatial modulation of the quadrupole field and that one Alfvén mode can mode convert efficiently to the other through the process analogous to the resonant parametric mode coupling. The present analyses on the mode conversion of Alfvén waves can reasonably explain the experimental results on the polarization reversal of Alfvén waves in GAMMA 10 and validate the mode-conversion mechanism proposed in Ref. 1.

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