Squeezing in the Self-Pulsing Domain

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We study analytically the squeezing spectrum of second-harmonic generation in the self-pulsing regime. We prove that squeezing is still defined in the presence of a limit cycle. When the input field exceeds the self-pulsing threshold, the intensity spectrum remains smaller than the shot-noise limit in a frequency domain around the self-pulsing frequency.

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It has been known for some time that squeezing can be enhanced near a bifurcation point.[1] This property has been recently demonstrated for a class of nonlinear optical systems[2] and is especially relevant for parametric processes inside a resonant cavity. For instance, in subharmonic generation on resonance, perfect squeezing is obtained at the bifurcation point corresponding to the transition from the amplifier regime to the oscillator regime.[1(a)] On both sides of the bifurcation point, the squeezing is large, though of a different nature. Below the bifurcation, a squeezed vacuum is produced,[1(a),3] while above the bifurcation, the subharmonic field displays squeezed phase fluctuations.[1] In the case of subharmonic generation, the bifurcation point corresponds to a transition between steady states. Both regimes are characterized by time-independent intensities. In second-harmonic generation (SHG), a different kind of bifurcation occurs (known as Hopf bifurcation), which connects a steady state to a time-periodic solution.[4] It has been shown that perfect intensity squeezing can be reached when approaching the Hopf bifurcation from below.[1(a)]

Most nonlinear optical systems in resonant cavities display such Hopf bifurcations. The purpose of this Letter is to calculate analytically the squeezing spectrum above a Hopf bifurcation, in the self-pulsing regime. To our knowledge, no theoretical result has been published on squeezing of nonsteady states (such as stable periodic or quasiperiodic solutions) with constant input. In this line of temporal problems, the parametric amplifier submitted to a periodically pulsed input field has been considered by Yurke *et al.*[5] and it was demonstrated that squeezing can still be defined. This has been confirmed experimentally first by Slusher *et al.*[6] The problem that we analyze in this Letter is different since the input field amplitude remains constant while both output electric fields display stable spontaneous amplitude and phase modulations. Thus it is nonlinear dynamics which is the cause of the time periodicity in the response of the system. In this first analytic approach to the problem, we shall consider SHG because its analytical solution of the deterministic problem in the self-pulsing regime is known and simple to handle.

There exist two main methods for calculating the squeezing spectra of nonlinear optical systems. The first one is an extension by Collett and Gardiner[7] of the fluctuation-dissipation theorem for nonlinear dissipative quantum systems. It was used by Collett and Walls[1(a)] who published the first theoretical derivation of the squeezing spectrum in SHG below the Hopf bifurcation. The second method is the semiclassical method of Reynaud and co-workers.[8,9] It leads to evolution equations for the field fluctuations linearized around a solution of the deterministic semiclassical equations. The fields outside the cavity are expressed in terms of the fields inside the cavity, using classical reflection-transmission equations. This allows the determination of the output fluctuations by regarding all fluctuations as driven by classical random fields incident to the mirrors. Although all previous applications of this method involved fluctuations around steady states, it is easily generalized to deal with fluctuations around a periodic solution, as long as the fluctuations remain small compared to the mean fields. We will use this method to analyze the squeezing spectrum in SHG in the self-pulsing domain.

The semiclassical equations of SHG are[4]

$$R_1' = -\gamma R_1 + R_1^* R_2 + E, \quad R_2' = -R_2 - R_1^2 \qquad (1)$$

for the scaled complex electric fields of the fundamental mode (R_1) and of the second harmonic (R_2), with E being the scaled pump field and γ the ratio of the two cavity-field decay rates. For the sake of simplicity, we shall only consider the good converter limit $\gamma \to 0$. In this limit the steady-state solutions $R_1 = E^{1/3}$ and $R_2 = -E^{2/3}$ are stable for $0 \le E \le E_H \equiv 1$. The critical point E_H is a Hopf bifurcation of the two field phases out of which a stable periodic solution emerges.[4] Near but above the Hopf bifurcation, it has been proved that the solution takes the form[10]

$$R_1 = 1 + i\epsilon(e^{i\tau} + e^{-i\tau}) + \epsilon^2 \left(\frac{2+i}{1-4i} e^{2i\tau} + \frac{2-i}{1+4i} e^{-2i\tau} + \frac{33}{34} \right) + O(\epsilon^3),$$
$$R_2 = -1 + i\epsilon[(i-1)e^{i\tau} - (i+1)e^{-i\tau}] + \epsilon^2 \left(\frac{3}{4i-1} e^{2i\tau} - \frac{3}{4i+1} e^{-2i\tau} + \frac{1}{17} \right) + O(\epsilon^3), \quad (2)$$

where $0 < \epsilon \ll 1$ defines the neighborhood of the bifurcation: $E = E_H + \frac{99}{34}\epsilon^2$. The time τ is scaled in such a way that the solutions are 2π periodic: $\tau = [1 + \frac{27}{17}\epsilon^2 + O(\epsilon^3)]t$, where t is the reduced time appearing in Eqs. (1).

Let $[R(\tau)]$ be a vector whose components are $\mathrm{Re}(R_1)$, $\mathrm{Re}(R_2)$, $\mathrm{Im}(R_1)$, and $\mathrm{Im}(R_2)$, respectively. Equation (1) can be written formally as

$$\frac{d}{d\tau}[R(\tau)] = N\{E, [R(\tau)]\}. \quad (3)$$

To determine the squeezing spectrum using the semiclassical method,[8,9] we assume that a weak noise source $\eta[\delta\alpha(\tau)^{\mathrm{in}}]$ is added to the constant input field,

$$[E(\tau)] = [E] + \eta T[\delta\alpha(\tau)^{\mathrm{in}}],$$
$$[E] = (E,0,0,0), \quad T = \mathrm{diag}(0,\sqrt{2},0,\sqrt{2}), \quad (4)$$

in the limit $\gamma \to 0$, where T is the diagonal transmission matrix through the mirrors and $|\eta| \ll 1$. We denote by $[\bar{R}(\tau)]$ the solution of the deterministic equation [Eq. (3)] obtained with the constant source $[E]$ and by $[\tilde{R}(\tau)]$ the solution of the stochastic equation [Eq. (3) when Eq. (4) is used]. We seek solutions of the form

$$[\tilde{R}(\tau)] = [\bar{R}(\tau)] + \eta[\delta\alpha(\tau)] + O(\eta^2), \quad (5)$$

above the self-pulsing threshold, i.e., $E \geq 1$. The noise vector inside the cavity satisfies the equation

$$\frac{d}{d\tau}[\delta\alpha(\tau)] = A(\tau)[\delta\alpha(\tau)] + T[\delta\alpha(\tau)^{\mathrm{in}}], \quad (6)$$

where the time-dependent matrix $A(\tau)$ is defined by

$$N\{E,[\tilde{R}(\tau)]\} = N\{E,[\bar{R}(\tau)]\} + \eta A\{[\bar{R}(\tau)]\}[\delta\alpha(\tau)] + O(\eta^2).$$

The noises outside the cavity are obtained using the reflection-transmission relations

$$[\delta\alpha(\tau)^{\mathrm{out}}] = T[\delta\alpha(\tau)] - [\delta\alpha(\tau)^{\mathrm{in}}], \quad (7)$$

and we assume that the input noises are Gaussian white noises. From Eq. (5), we obtain

$$[\tilde{R}(\tau)^{\mathrm{out}}] = [\bar{R}(\tau)^{\mathrm{out}}] + \eta[\delta\alpha(\tau)^{\mathrm{out}}] + O(\eta^2),$$

where $[\bar{R}(\tau)^{\mathrm{out}}] = [\bar{R}(\tau)]\sqrt{2}$ in the limit $\gamma \to 0$. With these solutions, we define the intensity of the second-harmonic field,

$$I(\tau) \equiv |\tilde{R}_2(\tau)^{\mathrm{out}}|^2 = |\bar{R}_2(\tau)^{\mathrm{out}}|^2 + \eta\delta I(\tau).$$

The squeezing spectrum is then calculated in the frequency domain. The amplitude spectrum outside the cavity is defined as $S_a(\omega) \equiv |\delta B(\omega)|^2/4$, with $\delta B(\tau) = \delta R_2(\tau) + \mathrm{c.c.}$ being the correction to the average amplitude. The spectrum is defined in this way so that the shot-noise level is equal to unity. Both $[\delta\alpha(\omega)^{\mathrm{out}}]$ and $[\bar{R}(\omega)^{\mathrm{out}}]$ are expanded in powers of ϵ since the periodic solutions (2) which we use are also expanded in such series. Note that in Eq. (7), $[\delta\alpha(\tau)^{\mathrm{in}}]$ is independent of ϵ. The amplitude-squeezing spectrum has therefore been calculated perturbatively:

$$S_a(\omega,\epsilon) = S_0(\omega) + \epsilon^2 S_{a,2}(\omega) + O(\epsilon^4), \quad (8)$$

$$S_0(\omega) = (\omega^2-1)^2/(\omega^4 - 2\omega^2 + 9), \quad (9)$$

$$S_{a,2}(\omega) = \frac{16}{17} \frac{11\,016 + 4998\omega^2 - 6881\omega^4 + 3870\omega^6 - 852\omega^8 + 89\omega^{10}}{(\omega-2)^2(\omega+2)^2(\omega^4 - 2\omega^2 + 9)^2\omega^2}. \quad (10)$$

The functions $S_0(\omega)$ and $S_{a,2}(\omega)$ are displayed in Figs. 1 and 2, respectively. The function $S_0(\omega)$ is the spectrum at threshold already known from below-threshold calculations.[1(a)] It lies entirely below the shot-noise limit and vanishes for $\omega = 1$. The correction $S_{a,2}(\omega)$ diverges for $\omega = 0$ and 2, and vanishes at infinity. Divergences at all harmonics of the self-pulsing frequency are expected in the squeezing spectra of a periodic solution. The interesting feature, however, is that we do not find all of these singularities in *both* the amplitude and the phase (imaginary part of the fluctuations) spectra. On the contrary, the amplitude spectrum displays divergences only at even multiples of the self-pulsing frequency, whereas divergences at odd multiples of the self-pulsing frequency will occur in the phase spectrum. However, this separation is not expected to remain out of resonance.

Let us now calculate the intensity-squeezing spectrum that is the most easily measurable function in this problem. When dealing with time-dependent solutions, the intensity- and the amplitude-squeezing spectra are no

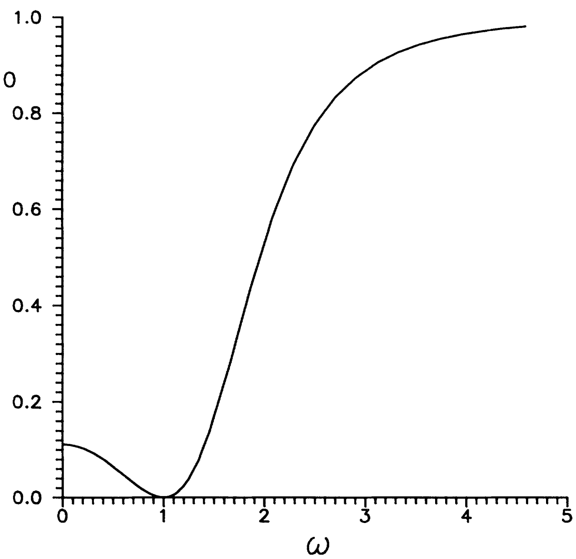

FIG. 1. Squeezing spectrum at the Hopf threshold.

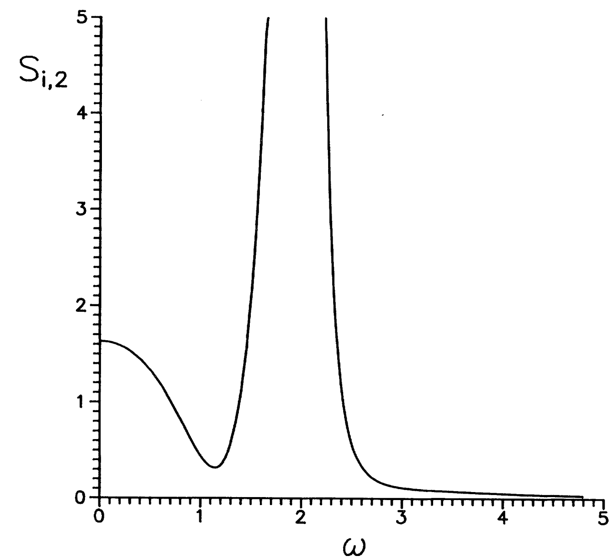

FIG. 3. First correction to the intensity-squeezing spectrum.

longer proportional to each other. Since $R_2(\tau)$ is periodic, its sidebands couple noise contributions at frequency ω with noise contributions at frequencies displaced by an integer multiple of the self-pulsing frequency. Indeed, the intensity spectrum is defined as $S_i(\omega) \equiv |\delta I(\omega)|^2/4\theta$, where $\delta I(\omega)$ is the Fourier transform of $\bar{R}_2(\tau)^{\rm out}\delta R_2^*(\tau)+$c.c. and $\theta=1+66\epsilon^2/17+O(\epsilon^4)$ is the average intensity of the second-harmonic mode. The intensity spectrum is $S_i(\omega,\epsilon)=S_0(\omega)+\epsilon^2 S_{i,2}(\omega)+O(\epsilon^4)$. At threshold, $S_i(\omega)=S_0(\omega)$ is given by (9). Its correction is

$$S_{i,2}(\omega)=\frac{16}{17}\frac{2244-3583\omega^2+1932\omega^4-342\omega^6+21\omega^8}{(\omega-2)^2(\omega+2)^2(\omega^4-2\omega^2+9)^2}. \tag{11}$$

This correction is displayed in Fig. 3. It has a minimum at $\omega \simeq 1.14$ with $S_2(1.14)\simeq 0.31$. The function $S_2(\omega)$ has three remarkable values: $S_2(0)=\frac{44}{27}\simeq 1.63$, $S_2(1)=\frac{4}{9}\simeq 0.44$, and $S_2(\infty)=0$. The main feature of $S_2(\omega)$ is a divergence for $\omega=2$ while the divergence at zero frequency in the amplitude spectrum does not show up here.

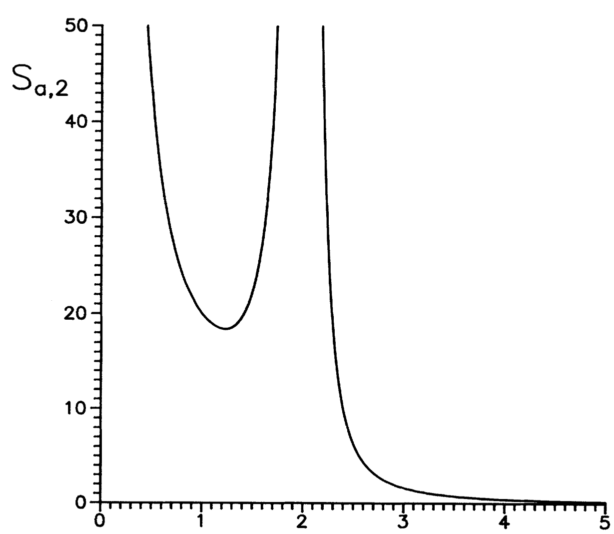

FIG. 2. First correction to the amplitude-squeezing spectrum.

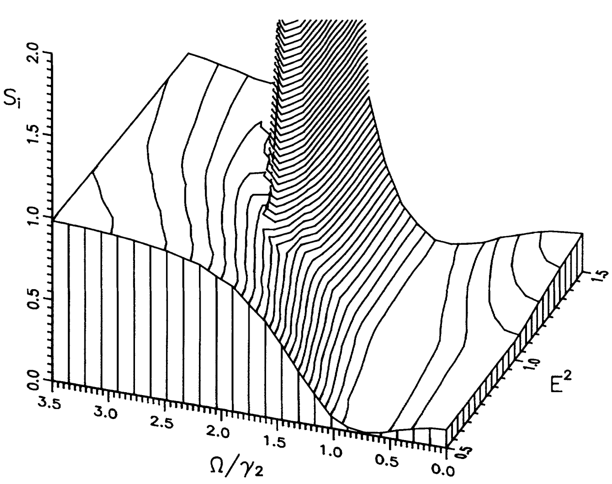

FIG. 4. Intensity squeezing vs both the reduced frequency Ω/γ_2 and the input intensity E^2.

It should be pointed out that the frequency ω is related to the time τ and is therefore a function of ϵ. The frequency which is measured experimentally is $\Omega = \gamma_2 \omega \times [1 + \frac{6}{11}(E - E_H)]$. Figure 4 displays the variation of the intensity noise spectrum as a function of Ω/γ_2 and of the pump intensity E^2 when the self-pulsing threshold is crossed. Apart from the large noise occurring around the frequency $\omega = 2$, the noise variation is smooth around the bifurcation. As the correction $S_2(\omega)$ is positive, the squeezing is reduced for all frequencies. The best squeezing still occurs in the vicinity of the self-pulsing frequency and remains very good even far from threshold.

In conclusion, we have shown that squeezing can still be defined in a self-pulsing regime although a distinction has to be made between intensity and amplitude spectra. We have extended the analysis of the squeezing across a steady bifurcation to the case of a Hopf bifurcation and given an analytic expression for the quantum noise. The principle of our calculation is by no means limited to SHG but can be applied to other nonlinear systems which also display Hopf bifurcations.

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[1](a) M. J. Collett and D. F. Walls, Phys. Rev. A **32**, 2887 (1985); (b) C. Fabre, E. Giacobino, A. Heidmann, L. A. Lugiato, S. Reynaud, M. Vadacchino, and Wang Kaige, Quantum Opt. **2**, 159 (1990).

[2]L. A. Lugiato, P. Galatola, and L. M. Narducci, Opt. Commun. **76**, 276 (1990).

[3]B. Yurke, Phys. Rev. A **29**, 408 (1984).

[4]P. D. Drummond, K. J. McNeil, and D. F. Walls, Opt. Acta **27**, 321 (1980).

[5]B. Yurke, P. Grangier, R. E. Slusher, and M. J. Potasek, Phys. Rev. A **35**, 3586 (1987).

[6]R. E. Slusher, P. Grangier, A. LaPorta, B. Yurke, and M. J. Potasek, Phys. Rev. Lett. **59**, 2566 (1987).

[7]M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984); C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).

[8]S. Reynaud, C. Fabre, and E. Giacobino, J. Opt. Soc. Am. B **4**, 1520 (1987).

[9]S. Reynaud and A. Heidmann, Opt. Commun. **71**, 209 (1989).

[10]P. Mandel and T. Erneux, in *Laser Physics*, edited by J. D. Harvey and D. F. Walls, Lectures Notes in Physics Vol. 182 (Springer-Verlag, Heidelberg, 1983), p. 182.