Generic Scale Invariance and Roughening in Noisy Model Sandpiles and Other Driven Interfaces

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(Received 31 October 1990)

From symmetry arguments we construct a simple Langevin model to describe driven interfaces such as lattice sandpile models composed of discrete grains in the presence of white noise. The model exhibits generic scale invariance (or "self-organized criticality"), with calculable exponents in all dimensions. For spatial dimensions $1 < d \le 2$ it undergoes a roughening transition between two distinct phases with algebraic correlations. The transition is Kosterlitz-Thouless-like in $d=2$.

PACS numbers: 64.60.Ht, 05.40.+j, 05.60.+w, 46.10.+z

Correlations in finite-temperature equilibrium systems with short-range interactions typically decay exponentially with distance for generic parameter values, the scale of the decay being set by the correlation length. Only by tuning one or more parameters of the system to critical values can one achieve scale invariance, i.e., infinite correlation lengths and algebraic decays of correlations, which are the hallmark of continuous phase transitions.

It has become clear in the past decade,[1] however, that certain noisy nonequilibrium (or "driven") systems exhibit *generic* scale invariance—infinite correlation lengths and the concomitant algebraic decays for arbitrary parameter values. In fact, this behavior is now believed to occur in almost all noisy nonequilibrium systems with a conservation law.[2-4] Moreover, Bak, Tang, and Wiesenfeld[5] have discovered that certain essentially[6] deterministic nonequilibrium systems such as model sandpiles[7] manage to spontaneously organize themselves so as to produce responses to external perturbations which are likewise scale invariant in space and time for generic parameter values. This behavior, dubbed "self-organized criticality" (SOC), has been proposed[5] as the origin of the ubiquitous occurrence in nature of both fractal structures[8(a)] and $1/f$ noise.[8(b)]

In this paper we construct and study a continuum Langevin equation[9] for a sandpile problem in which one imagines noise, in the form of sand grains dropped upon or removed from the pile, acting constantly and without correlation. As usual, the Langevin model is assumed to represent the coarse-grained physics of the underlying microscopic problem, and so to capture faithfully the long-time, long-wavelength behavior. We build here on the pretty work of Hwa and Kardar (HK),[2] who first proposed and analyzed a Langevin model for sandpiles. The system treated here differs from theirs in two main respects: First, we impose an extra symmetry which we believe is a crucial feature of sandpile (and of many interface) problems. Second, we add terms to HK's continuum sandpile theory which represent the effects of the discrete lattice present in the microscopic model sandpiles typically studied. The resulting model bears strong resemblance to the dynamical sine-Gordon theory[10] which describes the dynamics of the rough and smooth phases, and of the roughening transition, in familiar equilibrium interface systems.[9] While the model is motivated by sandpiles, it should describe other driven lattice interface systems[11] which have the same symmetries.

Our main results follow: In any spatial dimension d the model exhibits, in accordance with the results of Ref. 3, SOC: Spatial and temporal correlations decay algebraically. For $d>4$, the upper critical dimension, all nonlinearities are irrelevant, the theory reduces to a noisy diffusion equation, and the exponents describing the behavior of correlations assume the trivial mean-field (or Gaussian) values. For $2<d<4$ the linear theory is unstable, and the long-distance behavior is controlled by the strong-coupling fixed point analyzed by HK.[2] (Note that, like ordinary equilibrium interfaces,[10] the surface of the sandpile is, for all $d>2$, *smooth*; i.e., the surface width remains finite as the lateral size of the sandpile goes to infinity. Thus the system spontaneously breaks translational invariance.) For $1<d\le 2$, both the Gaussian and the (strong coupling) HK fixed points are stable, and have separate basins of attraction in the parameter space. These fixed points respectively describe *rough* and smooth phases for this range of dimensions; their basins of attraction are separated by a critical surface controlled by a critical roughening fixed point. For $d=1$ the stable Gaussian fixed point most likely attracts the entire parameter space.

These conclusions are obtained by application of renormalization-group (RG) methods to the Langevin model appropriate to the coarse-grained description of noisy sandpiles. In constructing such a model, we follow HK in picking out one particular (the parallel) direction in which the sand actually slides; the other $(d-1)$ (perpendicular) directions correspond to motions around the pile at constant height, and are assumed equivalent. (Obviously one could just as well take two sliding directions, say, and obtain different quantitative results, but with little conceptual change.) It remains only to identify the correct symmetry of the system to define the model fully. HK proposed the following equation for a continuum sandpile without discrete lattice structure:

$$\partial h/\partial t = \nu_\parallel \nabla_\parallel^2 h + \nu_\perp \nabla_\perp^2 h - \lambda \nabla_\parallel h^2 + \eta ; \qquad (1)$$

here $h(x_\parallel,x_\perp,t)$ represents the height of the sandpile at the point $(x_\parallel,x_\perp)$ and time t, measured from the average (linear) profile of the sandpile, and η is a random noise term with Gaussian (white) correlations:

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = D\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\,.$$

The noise is intended to represent grains of sand added to or removed from the pile at random, and so *does not* conserve the total amount of sand. (HK also considered a model with *conserving* noise.[2]) Note too that the deterministic part of model (1) conserves the field $h(\mathbf{x},t)$, consistent with real sandpiles wherein sand can leave the system only by sliding off a boundary. The model allows for different diffusion constants, $\nu_\parallel$ and $\nu_\perp$, in the parallel and perpendicular directions; λ is a coupling constant. It also incorporates translation invariance in $\mathbf{x}$, rotational invariance in the $(d-1)$ perpendicular directions, and invariance under the transformation $h\to -h$, $x_\parallel \to -x_\parallel$, representing "particle-hole" symmetry; i.e., a mound of sand sliding down the pile is equivalent to a depression sliding up the pile. This last symmetry prohibits the linear drift term $\partial h/\partial x_\parallel$ from appearing on the right-hand side of (1). It is not, however, a symmetry which is obeyed by generic dynamical rules for model sandpiles, though one can certainly construct specific sandpile rules which satisfy it. [Even without this symmetry, one can eliminate the $\partial h/\partial x_\parallel$ term by the Galilean transformation: $h(x_\parallel,t)\to h'(x_\parallel,t)\equiv h(x_\parallel - t,t)$.] Though all terms consistent with these symmetries should appear on the right-hand side of (1), HK showed that only the ones displayed in (1) are *relevant* under the RG: The others do not affect the long-distance, long-time behavior.

An important symmetry of sandpile and other interface systems missing from Eq. (1) is invariance of the equation under uniform translations: $h(\mathbf{x},t)\to h(\mathbf{x},t)+c$, for arbitrary constants c. This symmetry [violated by the nonlinear term of (1)] reflects the insensitivity of the system dynamics to anything except local height differences, i.e., gradients of h. While the existence of an equilibrium profile from which one measures heights suggests that one is not free to translate the pile arbitrarily, the position of this profile is determined by boundary conditions, and so should not affect the local dynamics. For example, a sandpile (of length L) can be created by imposing the boundary conditions $h(x_\parallel = L)=0$ and $\partial h(x_\parallel=0)/\partial x_\parallel = 0$ (which simulates the effect of an arbitrarily high wall at $x_\parallel=0$). Changing the condition at $x_\parallel = L$ to $h(x_\parallel=L)=h_L$, for some $h_L\neq 0$, while leaving the condition at $x_\parallel=0$ unaltered, creates a pile identical to the original but translated uniformly upward by h_L. The local dynamics of the two piles must, however, be identical, so that only gradients of h can appear in (1). The lowest-order nonlinear term consistent with this extra requirement has the form $\nabla_\parallel((\nabla h)^2)$. It is simple to show that this, and all other symmetry-allowed terms, are irrelevant under the RG. Thus the long-distance, long-time behavior is correctly given by the linear terms of (1), and so is not terribly exciting, though it does exhibit generic scale invariance, or SOC.

A more interesting theory—one that can be directly tested by computer simulation on discrete models—is obtained by the incorporation of a lattice structure. One imagines that the sandpile consists of a regular lattice of columns of sand, each column being a vertical stack of discrete grains of identical (say, unit) size. Thus the Langevin equation defining the model must be invariant under the transformation $h\to h+c$ not for arbitrary c, but only for integer c's. The coarse-grained equation consistent with this (and the earlier) symmetries is[12]

$$\partial h/\partial t = \nu_\parallel \nabla_\parallel^2 h + \nu_\perp \nabla_\perp^2 h - \lambda\nabla_\parallel \cos(2\pi h) + \eta\,. \qquad (2)$$

Obviously, terms of the form $\nabla_\parallel\cos(2\pi n h)$ for all integers n are also allowed by symmetry. As in the ordinary equilibrium sine-Gordon theory,[10] these higher harmonics turn out to be less relevant under RG transformation, and so need not be considered. The cosine term in (2) is the most relevant symmetry-allowed nonlinear operator. Note that (2) describes the long-distance and long-time properties not only of noisy sandpile models, but of any discrete nonequilibrium system which has $h\to h+n$ invariance and the other symmetries described above, and is driven preferentially in a particular direction.

Standard RG methods allow the analysis of the lattice model (2). For dimensions d close to 2, i.e., $d=2+\varepsilon$, the lowest-order recursion relations are

$$\partial\lambda/\partial l = \varepsilon\lambda - \lambda\Delta\,, \qquad (3a)$$

$$\partial\Delta/\partial l = -\varepsilon(1+\Delta) - B\lambda^2\,, \qquad (3b)$$

where B is a positive constant. These equations are the leading terms of a systematic expansion in the three small parameters λ, ε, and $\Delta \equiv \pi D/2(\nu_\parallel\nu_\perp)^{1/2}-1$; $\nu_\perp$ and the noise strength D do not renormalize.[2]

Equations (3) with $\nu_\parallel = \nu_\perp = \nu$ are virtually identical to the recursion relations obtained for the time-dependent Ginzburg-Landau equation,[10]

$$\partial h/\partial t = \nu\nabla^2 h - \lambda\sin(2\pi h) + \eta\,, \qquad (4)$$

describing the dynamics of the roughening transition in the equilibrium sine-Gordon interface theory.[10] This is not surprising, since the sandpile model (2) is also an interface problem, albeit a nonequilibrium one. Let us briefly review the interpretation of the RG flows (3) in the context of equilibrium roughening before discussing the sandpile problem: In $d=2$, Eqs. (3) give rise to the familiar Kosterlitz-Thouless[13] flow diagram [Fig. 1(a)]. This consists of a stable (Gaussian) fixed line at $\lambda=0$ for $\Delta>0$, with a corresponding basin of attraction comprising the wedge roughly described by $0\le\lambda<\Delta$. Points

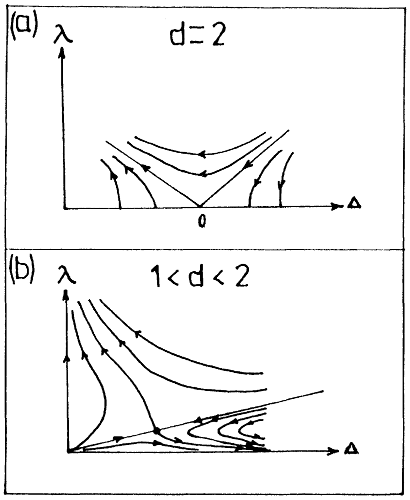

FIG. 1. RG flow diagrams for the recursion relations (3) for (a) $d=2$ and (b) $1<d<2$.

with $\lambda>0$ not lying in this basin flow off, as indicated in Fig. 1(a), to a strong-coupling fixed point at large λ and negative Δ (i.e., large v), beyond the range of validity of the recursion relations. It is well known[10,13] that the basin of attraction of the fixed line corresponds to the *rough* (high temperature) phase of the system: Initial parameter values in this basin give rise [as is seen by setting $\lambda=0$ in (4)] to algebraic decay of correlations with mean-field, or Gaussian, exponents; e.g., the dynamical exponent z is 2, the average width w of the interface, defined by $w^2 \equiv \langle h(\mathbf{x},t)^2 \rangle$, diverges logarithmically with the linear size L of the system, etc. On the other hand, the locus of points which run off to strong coupling define the *smooth* phase of the system, characterized by a w which remains finite as L diverges, and exponential decays of correlations.[10,13] One can infer this by noting that the RG flows to large λ and large v imply both a growing amplitude of the cosine potential and an increasing energy cost for fluctuations in h. This tends to localize the field h in one of the wells, thereby spontaneously breaking the $h \to h+n$ symmetry of (4), and producing a phase wherein $h(\mathbf{x})$ fluctuates only modestly about the bottom of the chosen well ($h=0$, say), i.e., the smooth (low temperature) phase. For large λ and v one can thus expand the sine term in (4) about $h=0$, i.e., replace $\sin(2\pi h)$ by $2\pi h$; the essential features of the smooth phase (e.g., the exponential decays) emerge immediately. Finally, the Kosterlitz-Thouless transition[13] between the rough and smooth phases is controlled by the fixed point at $\lambda=\Delta=0$.

For $1<d<2$ this qualitative two-phase picture remains intact,[14] the main difference being that the flow diagram is slightly different [Fig. 1(b)], the rough phase being controlled by a fixed point at $\lambda=0$, $\Delta=\infty$, rather than a fixed line.[14] The exponents characterizing the rough phase and the critical roughening exponents also vary with d. Again, the runaway flows to large λ imply the existence of a smooth phase, though one cannot extend this conclusion down to $d=1$, where exact calculations[15] show that there is no smooth phase. Finally, for $d>2$ there is no stable Gaussian fixed point or line at $\lambda=0$, only runaways to large λ, consistent with the existence of only a smooth phase [10,13,14] for $d>2$.

We return now to the actual sandpile problem, which is described by essentially the same recursion relations and RG flows. For $1<d\leq 2$ there is still a Gaussian rough phase with power-law correlations and hence SOC, and mean-field exponents controlled by a fixed point (or line for $d=2$) at $\lambda=0$. Again, the runaway flows to large λ and $v_{\parallel}$ for all $d>1$ signal a spontaneous breaking of the $h \to h+n$ symmetry and the occurrence of a smooth phase. The nature of this phase can, again, be inferred by choosing the well centered about $h=0$, and expanding the $\nabla_{\parallel}\cos(2\pi h)$ term in (2) to lowest nontrivial order in h, thus obtaining $\nabla_{\parallel} h^2$ as the nonlinear term. The resulting model is then precisely that of Eq. (1), studied by HK,[2] who showed that it exhibits SOC, and computed its exponents for all $d>1$. In particular, the interfacial width w varies with L like $L^{-(1-d)/(7-d)}$ for large L, implying a *smooth* interface for $d>1$. This must be the case if our reasoning is to be consistent: In expanding the cosine term in (2) we assumed that the system is localized in a single well, i.e., has finite width fluctuations, just as in the equilibrium roughening model. Unlike in that model, however, where the smooth phase is characterized by *exponential* decays[10] of correlations, the HK smooth phase has the algebraic decays characteristic of SOC, consistent with our earlier general arguments[3] that any conserving system with nonconserving noise must exhibit SOC.

Thus we arrive at the results summarized earlier: For $1<d\leq 2$ the sandpile model has two power-law phases, one rough, with mean-field (Gaussian) exponents governed by fixed points at $\lambda=0$, and one smooth, with exponents given by HK's strong-coupling fixed point. For $d>2$, the Gaussian fixed point becomes unstable, and all trajectories run off to the smooth, algebraic phase controlled by the HK fixed point. For $d>4$, the upper critical dimension for HK's model, the smooth phase has Gaussian exponents. The roughening transition connecting the rough and smooth phases behaves the same way as its analog for equilibrium roughening: Kosterlitz-Thouless-like[13] in $d=2$, and power-law-like[14] for $1<d<2$.

A remaining uncertainty is the behavior at $d=1$. The Gaussian rough phase controlled by the $\lambda=0$ fixed point is present, but the meaning of the runaway to strong coupling is less clear. It seems unlikely that the strong-coupling phase is smooth, since that would imply a broken $h \to h+n$ symmetry which, in $d=1$ with nonzero noise, is extremely difficult to achieve.[16] If the strong-coupling phase is *rough*, then the manipulations whereby we derived HK's model as the strong-coupling limit of our own are invalid. To complicate the matter, HK's model has not yet been solved right at $d=1$. To us the most likely possibility, given how difficult phase transitions are to achieve in noisy $d=1$ systems, is that the Gaussian rough phase occupies the entire phase diagram. Though we cannot rule out the existence of a distinct, algebraic, strong-coupling rough phase at large λ, preliminary numerical calculations[17] on a discrete, driven interface model whose long-distance behavior should be described by Eq. (2) find only the Gaussian rough phase. Finally, note that one can also analyze model (2) in the presence of *conserving* noise. For $d \geq 2$, the upper critical dimension,[2] one finds only a smooth, Gaussian phase, whereas for $d=1$ there is, as argued in Ref. 3, only a smooth phase with exponential correlations.

We thank M. P. A. Fisher, T. Hwa, C. Jayaprakash, M. Kardar, J. Krug, and particularly T. Bohr for useful discussions, and the Aspen Center for Physics, where part of this work was performed.

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[6]Most of the systems studied to date as paradigms for SOC have had noise either only in the initial conditions [e.g., J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989)], or noise with strong long-range correlations in time. The model sandpiles, e.g., are typically constructed (see Ref. 5) by dropping a sand grain at a random position on the pile, waiting until any subsequent avalanches have terminated, and then dropping another grain. The fact that noise events (i.e., dropped grains) are not permitted during the avalanches produces the noise correlations. Since the distribution $D(T)$ of avalanche duration times T falls off for large T as $T^{-\alpha}$, where (Ref. 5) $\alpha < 2$, the average time, $\overline{T} \propto \int dT\, T D(T)$, between consecutive noise events diverges for systems in the thermodynamic limit, where there is no finite-size cutoff to $D(T)$ at large T. For $\alpha < 1$ the noise power spectrum also diverges at zero frequency.

[7]Experiments on real sandpiles show that piles larger than several inches in diameter do not, in fact, exhibit SOC, but rather relaxation oscillations: H. M. Jaeger, C.-H. Liu, and S. R. Nagel, Phys. Rev. Lett. **62**, 40 (1989); G. A. Held *et al.*, Phys. Rev. Lett. **65**, 1120 (1990). The principles illustrated by the models are interesting nonetheless.

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[11]See, e.g., K.-t. Leung, K. K. Mon, J. L. Valles, and R. K. P. Zia, Phys. Rev. Lett. **61**, 1744 (1988).

[12]In general, the average profile of a microscopic lattice sandpile need not be linear; it could have periodic or even quasiperiodic height variations superimposed on a linear background. Presumably, however, the deviations from linearity become smaller and smaller as one does coarse graining to longer and longer length scales; we therefore treat the profile as if it were linear.

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