Physical Model of Intermittency in Turbulence: Inertial-Range Non-Gaussian Statistics

Zhen-Su She and Steven A. Orszag
Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544
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We present a physical model to describe the equilibrium probability distribution function (PDF) of velocity differences across an inertial-range distance in 3D isotropic turbulence. The form of the non-Gaussian PDF agrees well with data from direct numerical simulations. It is shown that these PDFs obey a self-similar property, and the resulting inertial-range exponents of high-order velocity structure functions are in agreement with both experimental and numerical data. The model suggests a physical explanation for the phenomenon of intermittency and the nature of multifractality in fully developed turbulence, namely, local self-distortion of turbulent structures.

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It has been observed, both experimentally[1] and numerically,[2,3] that the statistics of fully developed turbulence becomes increasingly non-Gaussian towards small scales. These intermittency effects are, in particular, characterized by increasingly non-Gaussian statistics of velocity differences with decreasing distance. The deviation from Gaussian behavior may be described by the exponents θ_n of the velocity structure function, defined by $\langle \delta v_l^n \rangle \sim l^{\theta_n}$, where δv_l is a velocity difference over a scale l. In the absence of the intermittency effects, $\theta_n \sim (n/2)\theta_2$ according to the Kolmogorov 1941 (K41) theory of turbulence.[4] Experimental[5] and numerical[3] data from nearly isotropic turbulence have shown unambiguous evidence of strong deviations from the predictions of the K41 theory, implying the existence of strong intermittency effects in the inertial range of turbulence.

Over the past thirty years, various phenomenological approaches have been developed to describe such intermittency in turbulence (see a summary in Ref. 6). The earliest is the log-normal model of Kolmogorov and Obukhov,[7] which assumes strongly non-Gaussian statistics for the energy dissipation. The log-normal model, however, fails to describe high-order exponents when compared to both numerical and experimental data. A more recent generalization of the log-normal model[6] does not seem to overcome the basic difficulty. A more popular model is the so-called multifractal model, introduced by Parisi and Frisch,[8] which relates θ_n, through a Legendre transform, to a set of fractal dimensions of physical space structures having a certain characteristic exponent. Two particular multifractal models, called the random β model[9] and the p model,[10] seem to give a better fit to experimental data than the log-normal model.[11] However, neither the multifractal model nor the log-normal model addresses the physical mechanisms behind the phenomenon of intermittency, so that these models seem arbitrary.

In the present Letter, we attempt to construct a physical model which aims at quantitatively describing the non-Gaussian probability density function (PDF) of velocity differences across inertial-range distances. This model is generalized from the one reported previously by us,[12] which has proved very successful in accurately describing near-dissipation-range non-Gaussian statistics, namely, the PDF of transverse velocity gradients. This approach is based on a physical model of the dynamics of the Navier-Stokes equation, and thus has a clear dynamical picture. In particular, we can see the physical mechanism of intermittency: local self-distortion of turbulence structures in physical space. The resulting PDFs allow us to calculate the inertial-range exponents $\xi_n = \theta_n - (n/2)\theta_2$; the agreement with both experimental and numerical simulation results is excellent. When the exponents θ_n are interpreted in terms of multifractals through a Legendre transform,[8] the present model suggests the physics behind multiple exponents.

Denote the velocity difference across a distance l by δv_l, which will be referred to as the amplitude of eddies of size l. Consider the evolution of eddies δv_{l_g} with near-Gaussian statistics at some scale $l_g = l + \delta l$ slightly larger than l. According to the classical K41 local cascade picture, the eddies will be squeezed by the random background and the amplitude of these eddies will decrease with exponent $\frac{1}{2}\theta_2 = \frac{1}{3}$. Eddies that result from this background squeezing process, denoted by δv_l^0 when the eddy size reaches l, will still have Gaussian statistics with typical amplitude $\delta v_l^0 \sim \delta v_{l_g}(l/l_g)^{1/3}$. We call this kind of background interaction a mean-field approximation. This picture is also consistent with second-order closure theories, where the near-Gaussian statistics has a precise meaning. We assume that, in the Navier-Stokes dynamics, the mean-field K41 process is the most dominant one in the sense that most eddies of small to moderate amplitudes undergo such a cascade. However, there is an additional process associated with high-amplitude eddies (large δv_{l_g}, hereafter referred to as structures), namely, coherent self-distortion that leads to non-Gaussian behavior. We assume that such self-distortion may lead to a dynamical exponent h significantly smaller than $\frac{1}{3}$. Thus, for structures, we have

$$\delta v_l = \delta v_{l_g}\left(\frac{l}{l_g}\right)^h = \delta v_l^0\left(\frac{l_g}{l}\right)^{1/3-h} = \delta v_l^0 J^{1/3-h}. \quad (1)$$

Here we introduce the mapping function $J = l_g/l$ which describes the squeezing ratio of the length scale. Note that J is a mean-history mapping function, following the spirit of Kraichnan,[13] so that (1) is only *statistically* valid.

To be more specific, consider the Navier-Stokes equation for the velocity difference:

$$D_t\delta\mathbf{v}_l = -\delta\mathbf{v}_l\cdot\nabla\bar{\mathbf{v}} - \nabla\delta p + \nu\nabla^2\delta\mathbf{v}_l\,,$$

where $\bar{\mathbf{v}} = \frac{1}{2}[\mathbf{v}(\mathbf{x}) + \mathbf{v}(\mathbf{x}+\mathbf{r})]$, $D_t = \partial_t + \bar{\mathbf{v}}\cdot\nabla$, and $\delta\mathbf{v}_l = \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$. Staying in a frame of reference which moves with the mean flow $\bar{\mathbf{v}}$, $\delta\mathbf{v}_l$ will be stretched by the strain $-\nabla\bar{\mathbf{v}}$. Following the above assumption (see also Ref. 12), the stretching term (including the action of the pressure) may be separated into two parts: a self-stretching part with a stretching rate proportional to $\delta\mathbf{v}_l$ which acts on structures only, and a mean (constant) stretching part which models the mean-field interaction between small $\delta\mathbf{v}_l^0$ eddies.

After replacing molecular viscosity by eddy viscosity and normalizing, all terms in the Navier-Stokes equations are of order unity. On the basis of the stretching model discussed above, we can then write the simplified equation

$$D_t\delta v_l = \delta v_l^2 + \delta v_l - J^2\delta v_l\,, \quad (2')$$

where J^2 originates from the reduction of eddy size in the Laplacian (second-order) dissipation term. Then using (1), we obtain an equation for J:

$$D_t J = \frac{|\delta v_l|}{C} f\left(\frac{|\delta v_l^0|}{C}\right) J + J - J^3\,, \quad (2)$$

$$f(x) = 1 - e^{-x}.$$

The function $f(x)$ is introduced to account for the fact that self-stretching only acts on structures. With this factor $f(x)$, Eq. (2′) is generalized so that it also applies to small-amplitude eddies for which $J \approx 1$. The parameter C is a threshold amplitude beyond which self-stretching is important, chosen[12] as $C = 2\langle(\delta v_l^0)^2\rangle^{1/2}$.

We can now express the PDF of the turbulent velocity difference in terms of the Gaussian PDF of δv_l^0:

$$P(\delta v_l) = P(\delta v_l^0)\,\partial\delta v_l^0/\partial\delta v_l\,. \quad (3)$$

We have previously shown[12] that the equilibrium PDF, the steady solution of (1)–(3), depends critically on the parameter h: As h decreases, the PDF becomes increasingly non-Gaussian (flat). In the framework of K41, where $h = \frac{1}{3}$, it can be easily checked that $P(\delta v_l)$ remains Gaussian and intermittency effects are absent. Physically, h describes the strength of the local self-stretching of structures in physical space, and, conversely, the strength of nonlocal interactions in wave-number space. The total nonlocal contribution in wave-number space for structures of a given size $l \sim 1/k$ should increase with k, since the width of the wave-number band of nonlinear interactions with small wave numbers increases (the interaction with higher wave numbers contributes to eddy damping). Thus, as l decreases, we expect that the deviation of h from $\frac{1}{3}$ may be amplified.

In Fig. 1, we compare two equilibrium PDFs resulting from the present model with two values of h to the PDFs of lateral velocity differences at two different length scales obtained from direct numerical simulations of isotropic turbulence. The agreement is satisfactory. The comparison also shows that h is indeed a decreasing function for decreasing length scale.

Assuming that in general h is a decreasing function of the length scale l at a given (high) Reynolds number, we can calculate the inertial-range exponents ξ_n. An approximate analytical calculation of the normalized nth-order flatness yields, for relatively large n,

$$F_n(h) = \langle\delta v_l^n\rangle/\langle\delta v_l^2\rangle^{n/2} \approx (n/2)^{n/(5/3+h)}.$$

If we define $l(h) = e^{-\mu/(5/3+h)}$ for some parameter μ, we readily obtain

$$\xi_n = -(1/\mu)n\ln(\tfrac{1}{2}n)\,. \quad (4)$$

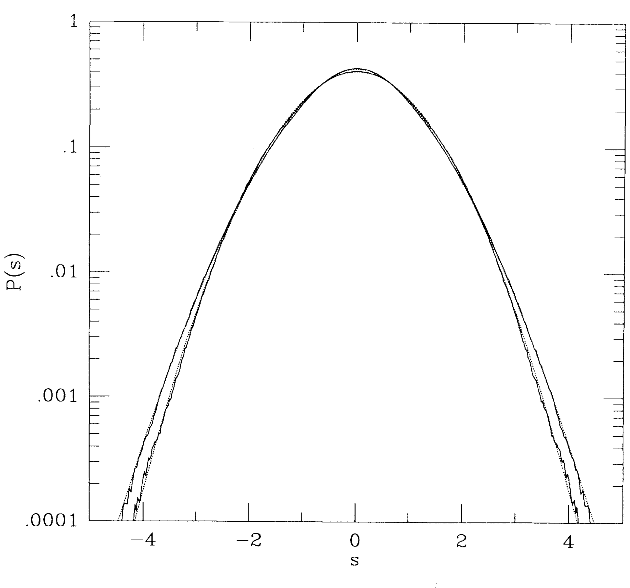

FIG. 1. The PDFs of the transverse velocity differences obtained from the steady solution of the model equations (1)–(3) (dotted lines) at two values of the parameter h, compared to the PDFs obtained from the direct numerical simulations of 3D isotropic turbulence of $R_\lambda \approx 77$ (solid lines) at two inertial-range distances l. For the inner lines, $h = 0.26$ and $l = 0.884$, and for the outer lines, $h = 0.03$ and $l = 0.393$.

We can then determine all exponents as a function of the single parameter μ. In this calculation, the second-order moment is evaluated using an asymptotic formula valid only for large n (the $\frac{1}{2}n$th power of the second-order moment is the normalization factor for the nth-order normalized moment). Within this approximation, the PDFs are exactly self-similar: For a given μ, all high-order moments vary with l in a power-law form. This result is verified by direct numerical solution of our model equations (1)-(3). We have computed $P_h(\delta v_l)$ numerically for all $-\frac{5}{3} < h \leq \frac{1}{3}$ and found $l(h)$ using the relation

$$\frac{\partial \log_{10}(l)}{\partial h} = \frac{1}{\xi_m}\frac{\partial \log_{10}F_m(l(h))}{\partial h} \tag{5}$$

for some, arbitrarily chosen, integer m. We then calculate all other exponents using the formula

$$\xi_n = \frac{\partial \log_{10}Fn(l(h))/\partial h}{\partial \log_{10}(l)/\partial h}\,. \tag{6}$$

Thus, ξ_n is determined up to one free parameter ξ_m, similar to μ in (4).

The degree of self-similarity depends on the constancy of $\xi_n(l)$ from (6) with respect to l. We have verified this constancy within an accuracy of 2% for all n.[14] This indicates that the exact solution $P_{l(h)}(\delta v_l)$ of (1)-(3) is also self-similar to a good approximation. Notice that although the solution (4) produces exponents very close to those obtained from (5) and (6) for a range of n, the trend $n\ln(\frac{1}{2}n)$ is not valid for very large n because the finite error in the analytical evaluation of the second-order moment, when taken to the $\frac{1}{2}n$th power, becomes very important.

In Fig. 2, we compare our exact exponents obtained by numerically integrating (1)-(3) and (5) and (6) to available numerical and experimental data. Data[3] from direct numerical simulation of isotropic turbulence are obtained at Taylor microscale Reynolds number $R_\lambda \approx 150$. The experimental data[5] are for a jet flow with $R_\lambda \approx 850$. The solid curve is the best fit by our model with an appropriate ξ_m. In both cases, as can be judged from Fig. 2, the agreement is very good, actually better than either the log-normal fit or the random-β-model fit.

The multifractal interpretation of intermittency effects in turbulence appears to be a valuable thermodynamical description. Extensive studies on the determination of the fractal-dimension spectrum in laboratory flows have been conducted at Yale University.[10] So far, there has not been any physical model explaining the nature of the multiple exponents. The present work gives a complete set of ξ_n which gives rise to the multifractality. Choosing $\theta_2 = \frac{2}{3}$ which is also a free parameter in our model, we can calculate the fractal-dimension spectrum $D(q)$ defined implicitly as $q = \partial\theta_n/\partial n$, $D(q) = 3 + nq - n\theta_2/2 - \xi_n$. The result is shown in Fig. 3, where the free parameter ξ_m is fixed at the value which gives the exponents in Fig. 2. At $q = \frac{1}{3}$, $D(q) = 3$, independent of ξ_m. When q decreases from $\frac{1}{3}$, the fractal dimension decreases below 3. To the limit of accuracy of the numerical integration, it seems that the exponent q approaches a limiting value $q_\infty \approx 0.09$, and the asymptotic dimension $D(q_\infty) \approx 1.4$. Note that q_∞ depends on θ_2 and $D(q_\infty)$ depends on ξ_m. We do not expect that they are

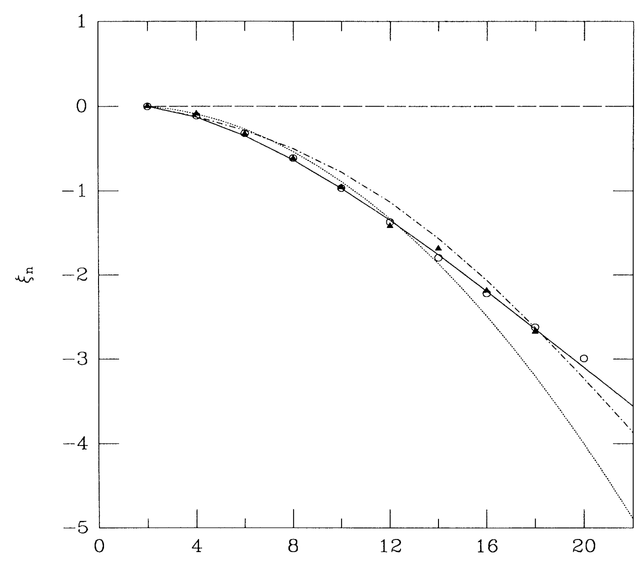

FIG. 2. The scaling exponents of dimensionless velocity structure functions ξ_n. ○, numerical data; ▲, experimental data; dashed line, K41 theory; dotted line, log-normal model; dash-dotted line, random β model; solid line, the present dynamical model.

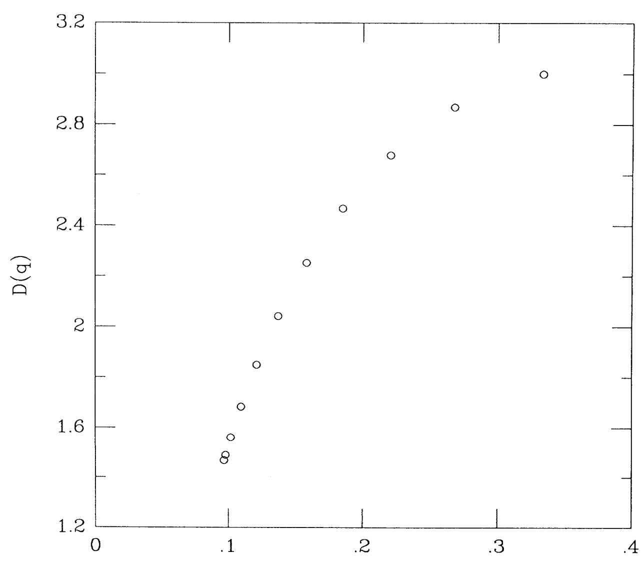

FIG. 3. The fractal-dimension spectrum derived from the present model for singularities of exponent $q \leq \frac{1}{3}$.

universal. However, it is clear that the physics behind the multiple exponents is the increasing importance of self-distortion of high-amplitude eddies which modifies the behavior of high-order moments differentially. Our present dynamical model can be regarded as a first step toward building a statistical-mechanical model for the thermodynamics of multifractality.

We have not addressed here the question of how intermittency effects modify the second-order exponent of the velocity structure function. We would like to point out, however, that in the framework of the present model, a $k^{-5/3}$ energy spectrum could be consistent with the presence of the intermittency effects.

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[14]The integration of (5) is started with a Gaussian distribution at $l=1$ (the integral scale). An interval of h is then directly mapped to an interval of l corresponding to the inertial range. The accuracy is determined by computing the rms relative deviation of ξ_n over two decades in l.