Renormalization Group for Diffusion in a Random Medium

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We develop a space-time renormalization-group method to study diffusion in a disordered medium. We prove rigorously that a random walk with transition probabilities given by a random matrix diffuses in a Brownian way for weak disorder if $d>2$. We also show that the disorder induces polynomial decay of velocity-velocity correlations.

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Random walks and polymer systems in a disordered environment have aroused much interest during recent years.[1] While many such systems can be adequately dealt with using perturbative renormalization-group (RG) methods, it has also become evident that nonperturbative effects can sometimes have drastic effects.[2-4] In this paper, we report on a new nonperturbative RG scheme developed for such systems and on a proof based on this scheme that diffusive behavior is stable under a wide class of local random perturbations. While the full proof appears in Ref. 5, we shall outline some of the main ideas here.

In a simple random walk, a particle jumps, at each time, from one lattice site to a neighboring one, with equal probability. The resulting motion diffusive, i.e., if the particle starts from the origin, its position after time t grows like $(Dt)^{1/2}$, where D, the diffusion constant, equals 1 in this simple example. Let us assume that one adds impurities to the system, and that these may influence the probabilities of jumping from site to site. Then these probabilities will be written $p(x,y)$ and will depend both on x, the point that the particle has already reached, and on y, the point to which it jumps. Since the locations of the impurities are random, we put a probability distribution on the set of $p(x,y)$'s and study the diffusive properties of the system for typical realizations of these transition probabilities.

More precisely, let us consider a random walk on $\mathbb{Z}^d$ described by the *transition probabilities* $p(x,y)>0$ from $x\in\mathbb{Z}^d$ to $y\in\mathbb{Z}^d$ satisfying

$$\sum_{y\in\mathbb{Z}^d} p(x,y)=1\,. \tag{1}$$

The probability of walking from 0 to x is then given by

$$P(x,t,p)=p^t(0,x)=\sum_\omega \mu_t(\omega)\delta_{\omega(t),x}\,, \tag{2}$$

where $\mu_t(\omega)$ is the probability of a walk starting from the origin:

$$\mu_t(\omega)=\prod_{i=1}^{t} p(\omega(i-1),\omega(i))\,. \tag{3}$$

A random walk in a *random environment* is a random walk where p is a *random matrix,* taken from some ensemble $\mathcal{P}$. Given a p, we may define the *diffusion constant*

$$D(p)=\lim_{t\to\infty} D(t,p)=\lim_{t\to\infty}\frac{1}{t}\sum_\omega \mu_t(\omega)\omega(t)^2\,. \tag{4}$$

That $0<D(p)<\infty$ means that the motion is diffusive. Actually, in many cases, including the ones below, $D(p)$ is deterministic, i.e., does not depend on p but only on the ensemble $\mathcal{P}$.

Let us describe an example of $\mathcal{P}$'s covered by our analysis. These are the "trapping" environments. We take

$$p(x,y)=\begin{cases}1/2d+b(x,y), & |x-y|=1,\\ 0, & |x-y|\neq 1\,,\end{cases} \tag{5}$$

with

$$\sum_y b(x,y)=0\,. \tag{6}$$

b is taken to satisfy the following properties. (1) Independence: We take $b(x,y)$ and $b(x',y')$ to be independent if $x\neq x'$; the environment is maximally *asymmetric.* (2) Isotropy: We demand that the distribution of b be isotropic in space, in particular, we have $\bar{b}=0$. (3) The generating function of b satisfies

$$\overline{e^{tb(x,y)}}\le e^{t^2\epsilon^2}\,, \tag{7}$$

for ϵ small.

Thus the randomness is weak. This, however, does not yet guarantee diffusive behavior, since such asymmetric environments may form *traps.* Therefore, we impose a condition on the probability that the $p(x,y)$'s are near zero: (4) For Γ large,

$$\mathrm{Prob}[p(x,y)\le(1/2d)e^{-N}]\le e^{-\Gamma N},\quad N\in\mathbb{N}\,. \tag{8}$$

Let us explain this condition. Consider the following configuration of p's: $p(x,y)\sim p(y,x)\sim 1-e^{-N}$, $p(x,z),p(y,z')\sim e^{-N}$, $z\neq y$, $z'\neq x$, and $p(z,x),p(z',y)\sim 1/(2d-1)$. The set $D=\{x,y\}$ is a trap: It is easy for the walk to enter D, but hard to leave. Note that such a configuration would be impossible in a symmetric model where $p(x,z)=p(z,x)$. The walk tends to stay in such a trap for a time $T\sim e^N$, and this could spoil the diffusive behavior if the average distance between such traps were $\sim T^{1/2}$. Since this distance is an inverse power of the probability in (8), we expect that, if the constant Γ in property 4 is small enough, the walk will not diffuse. What is less evident is that property 4 is a sufficient condition for diffusion. Indeed, traps can occur in our model in all scales since small, long-wavelength, drifts may "push" the walk into some regions of the lattice. The RG analysis is needed to study such multiscale phenomena.

The model (5) was shown by Sinai[2] to be subdiffusive in $d=1$ with $\langle x^2\rangle \sim C(\ln t)^4$. This behavior follows for all strengths of the disorder due to the traps. This should be contrasted with the symmetric case studied in Refs. 6–9, where normal diffusion holds in any dimension, provided that the disorder has very-short-range correlations (as here). Also, the continuous time version of the process on the lattice,

$$\partial_t P+\Delta P+\nabla\cdot(bP)\,,$$

has been shown to diffuse for all d when the vector field b is the gradient of white noise.[10] In this form, our model is analogous to

$$\overline{b_\mu(x)b_\nu(y)}\propto\delta_{\mu\nu}\delta(x-y)\,.$$

This model was studied in Refs. 11 and 12, using perturbation theory, which predicts diffusion in $d\geq 2$. Subsequently, it was shown in Ref. 4 that, allowing a b in (5) with correlations decaying exponentially (but not too fast), it is possible to produce subdiffusive walks in any d. The analysis of Ref. 13 shows that, for weak enough falloff, such environments correspond to small effective Γ in long-distance scales. Our rigorous results do not cover the whole range of values of Γ for which diffusion is expected to hold. See Refs. 1, 13, and 14 for a more detailed discussion of this question.

Let us define the *scaling limit* of the probability distribution, μ_t, of the walks as the measure $\nu^{(p)}$ on continuous paths in continuous time, on the interval [0,1], defined by

$$\nu^{(p)}=\lim_{t\to\infty}\mathcal{S}_t\mu_t^{(p)}\,, \tag{9}$$

where $\mathcal{S}_t$ is the operation of scaling of space by $t^{-1/2}$ and time by t^{-1}. We have

$$\int d\nu^{(p)}(\omega)\,\omega(1)^2=D(p)\,. \tag{10}$$

We prove the following.

Theorem.— Let $d>2$. Then the diffusion constant $D(p)$ takes a constant value $D\neq 0$ for almost all $p\in\mathcal{P}$. The scaling limit is given by the Wiener measure with diffusion constant D.

Remarks.— The convergence is for arbitrary bounded continuous functions in the path space. b in (5) is allowed to be long range, with rapid exponential falloff. The continuous time version of the walk can also be treated. We expect that our method can be extended to prove diffusion also in $d=2$. This would require a more detailed analysis of the RG.

Moreover, we show that the disorder produces long-range velocity-velocity correlations. Indeed, one has

$$\overline{\langle\dot\omega(0)\dot\omega(t)\rangle}\sim\mathcal{O}(\epsilon^2)/t^{d/2}$$

as $t\to\infty$.

To explain the RG, let us first recall the scale invariance of ordinary diffusion in $\mathbb{R}^d$. If $P(x,t)=\langle x|e^{t\Delta}|0\rangle$ is the transition probability in time t from 0 to x, then $P(x,t)=L^{-d}P(L^{-1}x,L^{-2}t)$. For our walk, the corresponding probability is given by (2). The RG we use is decimation in time and scaling in space. We fix the position of the walk in (2) at times that are multiples of L^2 and sum over the rest of the walk. Then we scale time by L^2 and space by L, to obtain a walk with new effective transition probabilities. Concretely, we write

$$\begin{aligned}P(x,t,p)&=p^t(0,x)=(p^{L^2})^{L^{-2}t}(0,x)\\&=L^{-d}p_1^{L^{-2}t}(0,L^{-1}x)=L^{-d}P\left(\frac{x}{L},\frac{t}{L^2},p_1\right),\end{aligned} \tag{11}$$

with

$$\begin{aligned}p_1(x,y)&=L^dp^{L^2}(Lx,Ly)\\&=L^d\sum_\omega\prod_{i=1}^{L^2}p(\omega(i-1),\omega(i))\equiv\mathcal{R}p(x,y)\,.\end{aligned} \tag{12}$$

p_1 is a transition probability *density* for walks on a finer lattice $L^{-1}\mathbb{Z}^d$ and so it is natural to use the convention for the matrix product $p_1p_1=\sum_z L^{-d}p_1(\cdot,z)p_1(z,\cdot)$, which explains the various powers of L.

$\mathcal{R}$ is the RG transformation: It maps an environment p to a new one, thereby implementing the scaling. In particular, for the diffusion constant, we have the identity

$$D(t,p)=D(L^{-2}t,\mathcal{R}p)=D(1,\mathcal{R}^np) \tag{13}$$

if $t=L^{2n}$. Hence the long-time behavior of the walk is understood if we can control the iteration of $\mathcal{R}$.

Since $\mathcal{R}$ scales the space by L^{-1}, as $n\to\infty$ we obtain a walk on $\mathcal{R}^d$. For such walks, $\mathcal{R}$ has a line of Gaussian fixed points given by

$$p_D^*(x,y)=(2\pi D/d)^{-d/2}e^{-d|x-y|^2/2D}\,, \tag{14}$$

which is just the transition probability density of the Wiener process with diffusion constant D.

Consider now p given by (5). Let us first compute $\mathcal{R}p$ perturbatively in b:

$$\mathcal{R}p=\mathcal{R}\bar p+(D\mathcal{R})_{\bar p}b+\cdots\,, \tag{15}$$

where $\bar{p}$, the mean of p, is given by the simple random walk. This first, deterministic term, on the right-hand side of (15) converges to the fixed point (14) upon interation, so we need to compute the variance of $(D\mathcal{R})_{\bar{p}}b$,

$$(D\mathcal{R})_{\bar{p}}b(x',y') = L^d \sum_{x,y} \sum_{t=0}^{L^2-1} \bar{p}^t(Lx'-x)\bar{p}^{L^2-t-1}(Ly'-y)b(x,y)\,. \tag{16}$$

Consider for example the case $x'=y'=0$. Then, we walk freely from 0 to x in time t and from x back to 0 in time L^2-t. The walk, in time $\le L^2$, predominantly stays in the L^d cube $\square$ at 0, hits a particular x with probability L^{-d} at a given time t, and 0 again with probability L^{-d}. Altogether, summing over L^2 times, we have

$$(D\mathcal{R})_{\bar{p}}b \sim CL^{2-d} \sum_{x\in\square} \sum_{|y-x|=1} K(x,y)b(x,y)\,, \tag{17}$$

with $|K(x,y)|<1$. Being the sum of L^d independent random variables, (17) seems to have variance $\simeq CL^{4-d}\epsilon^2$. This calculation suggests that the randomness becomes more relevant in longer scales if $d<4$. However, we have ignored a very important property of b, namely, (6): $\sum_y b(x,y)=0$. We may take advantage of this in (16) by replacing $\bar{p}^{L^2-t-1}(Ly'-y)$ there by

$$\bar{p}^{L^2-t-1}(Ly'-y) - \bar{p}^{L^2-t-1}(Ly'-x) \tag{18}$$

for the terms with $L^2-t-1\neq 0$. But, since $|y-x|=1$, (18) equals $\nabla_x \bar{p}^{L^2-t-1}(Ly'-x)$ and the extra derivative brings an extra inverse power of L due to the scaling $x\to Lx$. The variance of $D\mathcal{R}b$ thus seems to be $\sim L^{2-d}$ times the variance of b. The disorder is irrelevant in $d>2$.

The exact computation confirms the above analysis. However, there is a slight catch. The disorder $b(x,y)$ will not have small variance pointwise in x and y, since the scaling we are using in space brings ultraviolet singularities to the problem. These would be removed by a coarse graining in space, which, however, is not a natural thing to do, since it would spoil the Markov property of the walk. The UV singularities turn out to be harmless, in the sense that p's may be convoluted with each other, which is all one needs in a random walk.

One might think that the perturbative expansion of (15) is all we need to control the iteration of $\mathcal{R}$. However, this is not the case due to traps in higher scales, which render the perturbation theory divergent. Indeed, even if b in (15) was deterministically bounded and small, i.e., if we did not have any traps in the first scale, we see from the above computation of $D\mathcal{R}b$ that the latter can be as large as L times b. Hence, traps will be generated in higher scales. The theory will have two coupling constants, ϵ describing the small disorder and Γ describing the probability of traps. These will run as

$$\epsilon_n^2 = L^{(2-d)n}\epsilon^2\,, \tag{19}$$

and Γ enters in the escape probability from a unit cube $\square_x$ containing the point x,

$$\mathrm{Prob}\left[\int_{\square_x^c} p_n(x,y)dy < e^{-N}\right] < L^{-2n\Gamma}e^{-\Gamma N}\,. \tag{20}$$

To understand this flow of Γ, consider the typical event considered above, where there is a single trap, say at the origin, and no others within a distance $e^{N\Gamma}$. Hence, the probability of escaping from 0 is

$$\sum_{|y|=1} p(0,y) \sim e^{-N}\,. \tag{21}$$

In the next scale, in case no new traps are created at $\square_0$ ($=$1-cube in $L^{-1}\mathbb{Z}^d$ at 0), the probability of escaping from $\square_0$ is given in terms of the original walk, in time L^2,

$$\int_{\square_0^c} p_1(x,y)dy \sim \sum_{\omega} \mu_{L^2}(\omega) \sim L^2 e^{-N}\,, \tag{22}$$

since we have $O(L^2)$ times to exit from the trap. Thus the "trap strength" at 0 decreased to $N-2\ln L$. This, of course, is nothing but a recursive way to see that, once we "wait" long enough, the trap is harmless, as discussed above. Thus, ignoring new traps, and possible old traps nearby, the escape probability after n iterations is $e^{-N}L^{2n}$, i.e., $N_n = N-2n\ln L$, and

$$\mathrm{Prob}(N_n=N) \sim \mathrm{Prob}(N_0 = N+2n\ln L) \sim L^{-2n\Gamma}e^{-\Gamma N}\,,$$

yielding (20).

The full analysis is, of course, more involved. There may be many traps near our trap, but this naturally is even more unprobable and brings a small contribution to the above. There can also be new traps coming from the effective rates b_n. These, due to the running ϵ_n, come with a probability $e^{-cL^{(d-2)n}}$, which is an even smaller correction.

From the RG transformation (12) it is clear that, even if we started with p which is short range [like the nearest-neighbor walk (5)], the effective rates p_n are longer range. The true variance of b_n will be

$$\overline{b_n(x,y)b_n(x',y')} \sim \epsilon_n^2 e^{-|x-y|-|x'-y'|-|x-x'|}$$

(if $|y-y'|>1$). Since independence was a crucial element in the analysis of the linear RG, it is useful to localize the rates as

$$b_n = \sum_Y b_{nY}\,,$$

with b_{nY} and $b_{nY'}$ independent if $Y\cap Y'=\varnothing$. Here Y are unions of unit cubes in $\mathbb{R}^d$. b_Y collects terms in a resummed perturbation expansion from walks scattering from impurities in the set L^nY in the original scale.

By a resummed expansion we mean the following. The expansion (15) makes sense only if b is small. For b large, i.e., a possible trap, we do not expand. This defines a "trapping" region in the lattice where the bounds (20) are iterated. Eventually, as we saw above, the traps tend to become weak and can be incorporated in the perturbative analysis. This constitutes the resummation. At each scale, new traps are formed and some old ones are resummed. The picture of the effective environment is the following. We describe it by random variables b_{nY} with variance as above and the "trap density" variables $N(x)$ describing the amount of trap at x, in the sense of the bound (20). As $n \rightarrow \infty$, the traps and the b_n's disappear and the probabilities tend to the Gaussian fixed point (14) with probability 1.

Finally, the velocity-velocity correlation is dominated by the walks returning to the origin after time t, which implies the $t^{-d/2}$ behavior.

We believe that a similar RG scheme can be used in other diffusion problems, such as the classical motion of a particle in the presence of randomly located scatterers (the Lorentz gas and its lattice versions), or the Anderson tight-binding model for electrons in a random potential. Upon a suitable coarse graining, such problems can be related to random walks in a random medium. We hope to return to these questions in the future.

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