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New Class of Level Statistics in Quantum Systems with Unbounded Diffusion

T. Geisel, R. Ketzmerick, and G. Petschel

Institut für Theoretische Physik und Sonderforschungsbereich Nichtlineare Dynamik, Universität Frankfurt, D-6000 Frankfurt/Main 11, Federal Republic of Germany

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We point out a new class of level statistics where the level-spacing distribution follows an *inverse* power law $p(s) \sim s^{-\beta}$, with $\beta = \frac{3}{2}$. It is characteristic of level clustering rather than level repulsion and appears to be universal for systems exhibiting unbounded quantum diffusion on 1D lattices. A realization of this class is met in a model of Bloch electrons in a magnetic field, where we find a purely diffusive spread of wave packets without the quantum limitations known from chaotic systems like the kicked rotator.

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Level statistics is a well established tool for the study of quantum systems with a complex structure of excited states, e.g., systems that are chaotic in the classical limit.[1] Depending on symmetry properties of the Hamiltonian, one distinguishes three universality classes. Level repulsion causes a power-law behavior $p(s) \sim s^{\beta}$ of the probability density of the nearest-neighbor level spacings s, where $\beta = 1,2,4$ and $s \to 0$. The stiffness of a spectrum can be determined from a quantity $\Delta_3(L)$, which, e.g., grows logarithmically in L for systems belonging to the Gaussian orthogonal ensemble (GOE).[1] In the present Letter we extend this concept to a new class of systems which show some type of level clustering rather than level repulsion. We found a level-spacing distribution $p(s)$ which follows an *inverse* power law $p(s) \sim s^{-\beta}$, with $\beta = \frac{3}{2}$. The $\Delta_3(L)$ statistics for the spectral stiffness follows another power law $\Delta_3(L) \sim L^{\gamma}$, with $\gamma = 1.493 \pm 0.002$. A heuristic argument, whose validity is verified numerically, indicates that the exponent $\beta = \frac{3}{2}$ is universal for quantum systems with unbounded diffusion in one dimension.

As a physical example we consider the dynamics of electrons in a crystal lattice subject to a homogeneous magnetic field. We have previously treated the classical limit of this problem, which can be approached in lateral surface superlattices (LSSLs) on a semiconductor heterojunction.[2] There we found normal and anomalous diffusive motions caused by the chaotic dynamics of the particle. On the other hand, this is a quantum-mechanical system, where we may ask for the quantum analogs of the chaotic diffusion. This question was studied intensely for the kicked rotator, where the classical chaotic diffusion is mimicked by the quantum system only initially.[3,4] After a finite time, quantum interferences impose a finite bound on the diffusive growth of the variance (quantum limitations of diffusion).[5] In the present context we show that the inverse power law $p(s) \sim s^{-3/2}$ can be understood, if we assume *unlimited* diffusion of the quantum-mechanical wave packets. The assumption is confirmed by a numerical simulation exhibiting a purely linear (diffusive) growth of the variance. This strongly contrasts the behavior previously known from the kicked rotator.

We study the level statistics of Bloch electrons in a magnetic field B in the framework of the Peierls substitution, which leads to a discrete Schrödinger equation in a quasiperiodic potential (Harper's equation)[6–8]

$$\psi_{n+1} + \psi_{n-1} + \lambda \cos(2\pi n\sigma - \varphi_0)\psi_n = E\psi_n , \qquad (1)$$

where ψ_n is the wave function at site n and $\lambda = 2$. The dimensionless parameter $\sigma = a^2 eB/hc$ gives the number of flux quanta per unit cell of area a^2 and determines the incommensurability of the system. For comparison, we will also consider cases $\lambda \neq 2$, as it is known that $\lambda = 2$ is a critical case[9] separating a regime of extended states ($\lambda < 2$) from a regime of localized states ($\lambda > 2$) for irrational σ.[10] For $\lambda = 2$, the states are neither localized nor extended and the spectrum is a Cantor set.[11] Considering it as a perturbation, the incommensurate potential breaks translational symmetry, lifts the twofold degeneracy, and introduces a dense set of gaps into the tight-binding band[8] $E(k)$ (see, e.g., Fig. 1). For σ a Liouville number, the spectrum is singular continuous.[12]

At first sight it seems impossible to do level statistics on an uncountable set of levels such as the Cantor spectrum. We observe, however, that all energies are bounded (Fig. 1) and that one can count the number of energy gaps larger than some size s. By varying s we thus can obtain an integrated level-spacing distribution (ILSD) apart from normalization

$$p_{\rm int}(s) = \int_s^\infty p(s')ds' , \tag{2}$$

whose derivative $p(s) = -dp_{\rm int}/ds$ determines the probability density of level spacings s. These functions can be normalized by introducing a lower cutoff $s_0 > 0$. The level spacings s are normalized to mean spacing 1.

The levels and their spacings are obtained numerically with the use of transfer matrices $\mathbf{M}_1(n,E)$.[13] One can replace Eq. (1) by a matrix equation

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \mathbf{M}_1(n,E) \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} , \tag{3}$$

where

$$\mathbf{M}_1 = \begin{pmatrix} E - \lambda\cos(2\pi n\sigma - \varphi_0) & -1 \\ 1 & 0 \end{pmatrix} . \tag{4}$$

We approximate the irrational incommensurability σ by successive rational convergents of its continued-fraction expansion. For $\sigma = p/q$, the potential is periodic with period q. We thus analyze the matrix product

$$\mathbf{M}_q(E) = \prod_{n=0}^{q-1} \mathbf{M}_1(n,E) , \tag{5}$$

which transfers the states (ψ_0, ψ_{-1}) into the states (ψ_q, ψ_{q-1}). According to the Bloch theorem, $\psi_{n+q} = e^{ikq}\psi_n$ and thus $\mathbf{M}_q(E)$ has eigenvalues $e^{\pm ikq}$, i.e.,

$$\mathrm{Tr}\mathbf{M}_q(E) = 2\cos(kq) . \tag{6}$$

This leads to the condition $|\mathrm{Tr}\mathbf{M}_q(E)| \leq 2$, from which one can determine the allowed eigenvalues E of Eq. (1).[8] The eigenfunctions at sites $n = 0$ and -1 form the corresponding eigenvectors of $\mathbf{M}_q$. The eigenfunction at site m is obtained by multiplying with the matrix $\mathbf{M}_m(E)$.

Figure 1 illustrates the spectral changes, i.e., the allowed energies as a function of λ, for an approximant of the golden mean $\sigma_G = (\sqrt{5} - 1)/2$. Energy levels indicative of the localized regime on the right-hand side ($\lambda > 2$) turn into pronounced bands in the extended regime ($\lambda < 2$). As λ decreases towards $\lambda = 2$, the magnetic-field case which we will consider henceforth, the levels arrange in clusters and form a self-similar hierarchy. If we apply the concept of the spectral staircase function $N(E)$ of level statistics[1] to this spectrum, we obtain a complete devil's staircase. The integrated level-spacing distribution Eq. (2) is shown in Fig. 2 for $\lambda = 2$ and two different rational approximants σ of the golden mean σ_G. It clearly displays an inverse power law

$$p_{\rm int}(s) \sim s^{1-\beta} , \tag{7}$$

and thus the level-spacing distribution (LSD) behaves as

$$p(s) \sim s^{-\beta} , \tag{8}$$

where $\beta = 1.5009 \pm 0.0010$. This equation expresses the

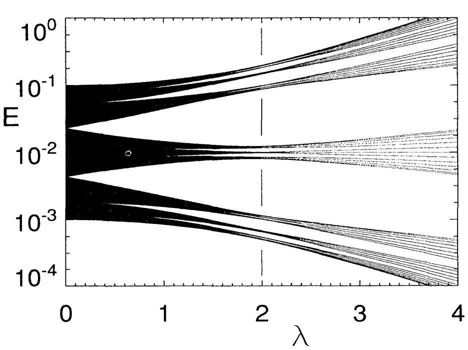

FIG. 1. Allowed energies as a function of the parameter λ for a rational approximant $\sigma = \frac{34}{55}$ of the golden mean. The magnetic-field case ($\lambda = 2$) is at the transition between regimes of extended states ($\lambda < 2$) and localized states ($\lambda > 2$) for incommensurate σ. As λ approaches $\lambda = 2$ from above, the levels arrange in clusters.

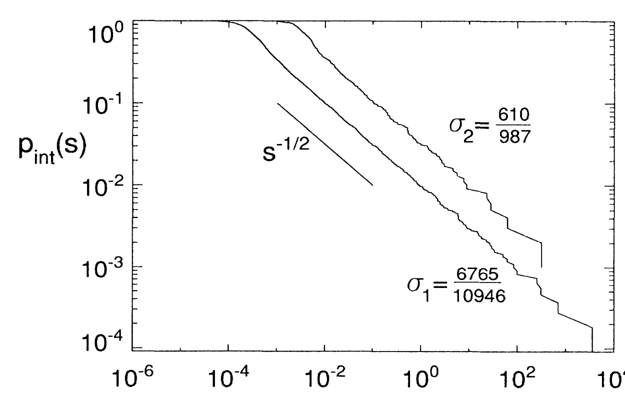

FIG. 2. Integrated level-spacing distribution ($\lambda = 2$) for two approximants of the golden mean displaying an *inverse* power law $p_{\rm int} \sim s^{1-\beta}$, with $\beta = 1.5009 \pm 0.0010$. As is seen the lower cutoff of the scaling region decreases for higher approximants.

self-similarity of the structure of gaps. Unfolding the spectrum by a smoothened spectral staircase as in other cases would not change the power law, as the spectral fluctuations remain self-similar on all scales. In Fig. 2 the ILSD levels off at a small value s_0, since for all rational approximants of σ_G the total number of gaps is finite. The cutoff s_0, however, can be shifted to arbitrarily small values for higher approximants. The LSD of Eq. (8) behaves very differently from Poisson, Wigner, and intermediate distributions found in other systems.[1] The increasing probability for smaller spacings indicates what we call level clustering (see also Fig. 1).[14] This property is more pronounced in another quantity,[1] the probability density $\mu(x)$ defined by the conditional probability of finding a level in $[x_0+x, x_0+x+dx]$, if there is a level at x_0 and no level in $]x_0, x_0+x[$. For a Poisson distribution, one has $\mu(x)=$const corresponding to independent level positions. For a Wigner distribution, $\mu(x)\sim x$ reflecting the repulsion of levels. In our case we have found $\mu(x)=(\beta-1)x^{-1}$ expressing a preference of clustering other levels in the vicinity of a given one. This property also affects the Δ_3 statistics of the spectrum shown in Fig. 3. We find that $\Delta_3(L)$ closely follows a power law $\Delta_3(L)\sim L^\gamma$, with $\gamma=1.48\pm0.06$, in clear contrast to a Poissonian spectrum $(L/15)$ and random-matrix theories $(\ln L)$.[1] The spectrum thus is even less rigid than a Poissonian spectrum.

Of course, one would like to understand what causes this new class of level statistics. In accordance with random-matrix theories, we have degenerate levels here that are split by the perturbation. In distinction, however, the degeneracy is not accidental, but systematically twofold (for states k and $-k$). The matrix elements of the perturbation therefore are not random, but are due to the quasiperiodic potential. Besides, neighboring splittings of levels are not independent, but affect each other as levels are nowhere isolated.[15] The exponent β of the LSD can be related to the fractal dimension D_0 of the spectrum. It is possible to show[16] that $D_0=\beta-1$.

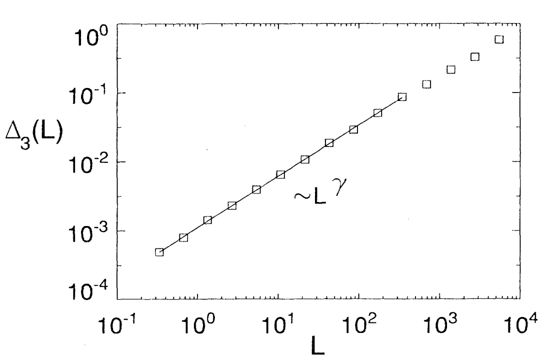

FIG. 3. Δ_3 statistics for $\lambda=2$ and $\sigma=6765/10946$. A least-squares fit (straight line) yields $\Delta_3(L)\sim L^\gamma$, with $\gamma=1.48\pm0.06$, in accordance with Eq. (9) ($\gamma=1+D_2=1.493\pm0.002$).

The numerical value of β appears to be largely independent of the incommensurability σ.[17] The fact that $\lambda=2$ is the critical point of the delocalization transition suggests that $\beta=\frac{3}{2}$ is a universal exponent. There are renormalization techniques for Eq. (1),[9,18,19] but local scaling properties (e.g., near $E=0$) are not sufficient to explain the global power law Eq. (8). In fact, it was found that the spectrum is a multifractal.[20] We can relate the number statistics[1] $n(L)$, which counts the number of levels in an interval of length L, to the multifractal scaling properties. For the moments of their distribution we show[21]

$$\langle n^q(L)\rangle \sim L^{1+(q-1)D_q}, \tag{9}$$

where D_q are the generalized dimensions. From Eq. (9) we obtain $\Delta_3(L)\sim L^\gamma$, with $\gamma=1+D_2$. A numerical determination of $D_2=0.493\pm0.002$ yields an improved value of γ consistent with Fig. 3 and shows that γ is different from $\frac{3}{2}$.

The global character of the exponent β asks for a global argument for its explanation. We can give a heuristic argument similar in spirit to arguments developed by Allen[22] and Chirikov, Izrailev, and Shepelyansky[4] for localization problems. We consider successive rational approximants $\sigma_i=p_i/q_i$ of the continued-fraction expansion of σ. If we want to resolve the spectrum with a finite resolution only, it suffices to confine the potential to a finite interval of length q_i. On this length scale the periodicity of the potential is not manifest and we may assume that a wave packet moves diffusively inside, i.e., $\langle x^2(t)\rangle \sim 2Dt$. The maximum distance q_i to be traveled defines a longest time scale $\tau\sim q_i^2/2D$ and a smallest energy difference between levels $s\sim\hbar/\tau$. The number of states living in the interval is $\sim q_i$ and thus determines the number of states with spacing $\Delta E \geq s$, whence $p_{\rm int}(s)\sim q_i \sim (2D\tau)^{1/2}=(2D\hbar)^{1/2}s^{-1/2}$. For a refined energy spectrum consider the next approximant p_{i+1}/q_{i+1}, where again the potential looks random within a period q_{i+1}. Repeating the argument yields the observed LSD Eq. (8) on all scales.

This argument reposes on the assumption of diffusion, to be repeated on all length scales. It suggests that the exponent $\beta=\frac{3}{2}$ is universal for systems showing unbounded quantum diffusion in one dimension. A more rigorous argument of Guarneri[23] concludes that the spectrum must be singular continuous and allow only values of $\beta \leq \frac{3}{2}$. The assumption, however, is not obvious in our case. In particular, one might also expect that the diffusive growth is nonlinear in time. This motivated us to analyze the time evolution of a wave packet $\phi(t)$ released at site 0, using the eigenenergies and eigenfunctions obtained above. Figure 4 shows the variance

$$\sigma^2(t)=\sum_{n=-q_i/2}^{q_i/2} n^2|\phi_n(t)|^2 \tag{10}$$

for three different values of λ in a finite lattice of

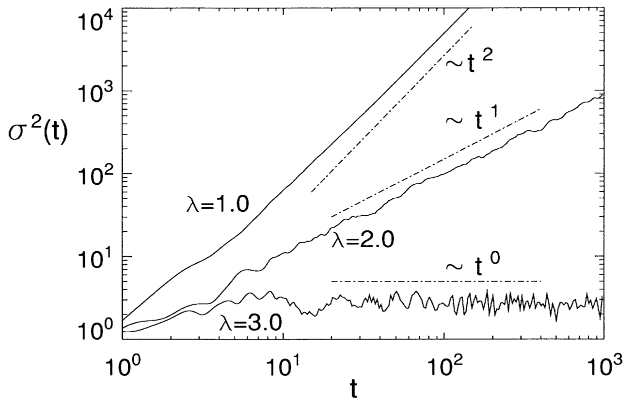

FIG. 4. Time evolution of the variance of a wave packet, Eq. (10). In the case $\lambda=2$ the spread is purely diffusive [$\sigma^2(t) \sim 2Dt$].

$q_i=987$ sites. The quadratic growth for $\lambda=1$ and the boundedness for $\lambda=3$ correspond to extended and localized eigenstates, respectively. In the critical case $\lambda=2$ we find clear-cut diffusion ($\sigma^2 \sim 2Dt$) with $D \approx \frac{1}{2}$. We mention that the wave packet reaches the boundary of the lattice of q_i sites in a finite time. We found that this time scales like q_i^2 and thus diffusion never stops for $q_i \rightarrow \infty$. In this way the validity of the heuristic argument is verified step by step and in particular the assumption of unlimited diffusion is confirmed.

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[15]The main reason for the nonapplicability of random-matrix theories is the absence of accidental degeneracies.

[16]In the box-counting method, integrating $sp(s)$ from l to infinity approximately gives the number of empty boxes times the box length l. This implies that the number of filled boxes scales like $l^{-(\beta-1)}$.

[17]Exceptions are the Liouville numbers, where we could not find an inverse-power-law behavior, whereas for a variety of irrational values of σ having small numbers in their continued-fraction expansion we always found $\beta=\frac{3}{2}$.

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