

Theory of Amplified Ferrimagnetic Echoes

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Amplification of echo signals in ordered magnetic materials was postulated by many authors in the middle sixties. However, theoretical estimates of amplification were invariably several orders of magnitude higher than observed in ferrimagnetic materials. We have reformulated the equation of motion to include the nonlocal dipole-dipole interaction to describe the dynamic response of an echo. Our estimates of amplification compare reasonably well with measured values.

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Ferrimagnetic echo phenomena were reported in cylinders and truncated spheres of yttrium iron garnet (YIG) crystals by Kaplan and co-workers^{1,2} and in single-crystal YIG films by Bucholtz, Webb, and Young.³ Echo experiments offer the possibility of a novel approach to performing important signal processing functions such as nondispersive time delay and pulse correlation in the frequency range below 10 GHz.⁴ With the demand of electronic technology advances there is also a renewed interest in the use of ferrimagnetic echo devices. Earlier theoretical work was successful in arguing for the presence of ferrimagnetic echoes and the possibilities for amplification.⁴⁻⁸ However, quantitative predictions of the amplification factor ranged between 1 and 2 orders of magnitude greater than observed. The purpose of this paper is to improve upon the predictability of the amplification factor in ferrimagnetic echo devices. The basic difference between previous and our theoretical formulations of ferrimagnetic echoes is as follows. In the former formulation⁸ the nonlinearity introduced in the equation of motion was due to only local field interactions. In our formulation we include the long-range dipole-dipole interaction in the nonlinear coupling term which is nonlocal in nature. Clearly, in the regime of very short delay times the two formulations predict the same result, since the time elapsed is too short to allow for long-range interactions to be effective. For example, in this limit we identify the parameter q , which was introduced *ad hoc* from earlier formulations,⁸ in terms of a physical quantity. However, for long delay times the previous formulations are simply not appropriate due to the exclusion of the long-range dipole-dipole interaction.

We consider a single-crystal YIG cylinder of rectangular cross section and infinite length. The dc field is applied along the cylindrical axis and a constant gradient of the field is assumed. These assumptions are purposely chosen to simulate the experiments presented in Refs. 1 and 2. Instead of linearizing the Gilbert equation we expand the equation to third order in \mathbf{m} and obtain

$$\dot{\mathbf{m}} = \hat{z} \times [(H_0 + Gz)\mathbf{m} - (1 - \mathbf{m} \cdot \mathbf{m}/2)(\mathbf{h}_m + \mathbf{h}_s) - \Lambda \nabla^2 \mathbf{m} + \lambda \dot{\mathbf{m}}]. \quad (1)$$

Here magnetization and fields have been normalized with respect to M_s , the time has been normalized with respect to $(\gamma M_s)^{-1}$, Λ is the exchange constant, λ is the

Gilbert damping constant normalized with respect to γM_s , and M_s and γ are, respectively, the saturation magnetization and gyromagnetic ratio. We denote \mathbf{h}_s as the externally applied rf field and \mathbf{h}_m as the dipolar field associated with the transverse magnetization \mathbf{m} , and \mathbf{m} is related to the total magnetization \mathbf{M} by the following relationship:

$$\mathbf{M} = M_z \hat{z} + \mathbf{m},$$

with

$$M_z \approx 1 - \mathbf{m} \cdot \mathbf{m}/2.$$

In Eq. (1), H_0 denotes the uniform dc nonspatially dependent internal field which is superimposed by a constant gradient field Gz with G being normalized with respect to M_s . In deriving Eq. (1) we have made the following assumptions. For long-wavelength excitation the exchange field is small and in YIG the damping field is small, so that only linear terms in these fields are kept in Eq. (1). Anisotropic fields are omitted in Eq. (1), since H_0 is applied along a cubic major axis. Also, for YIG, anisotropy fields are small compared to demagnetizing fields. We ignore the demagnetizing field in the z direction by assuming the sample to have a very long axial length. It can be argued that the nonlinearity due to the dc demagnetizing field, which is closely related to Suhl instability,⁹ will reduce echo gain and should be avoided. Further discussion in this regard will appear in a future paper. Finally, we have omitted the second-order term $(\mathbf{h}_m + \mathbf{h}_s)_z \mathbf{m}$ in Eq. (1). The reason is that, as we may verify from the following derivations, $(\mathbf{h}_m)_z$ and \mathbf{m} possess opposite polarities in x and y and will have vanishing effects on the equation of motion after they have been averaged over the sample's cross-sectional area. We assume \mathbf{h}_s to be a transverse field; hence, $(\mathbf{h}_s)_z$ is negligible by design in an experiment. We note that it is generally recognized that the nonlinearity must be of an odd order in order to generate an echo.⁴

Under the magnetostatic approximation the dipolar field can be written in the form¹⁰

$$\mathbf{h}_m = \nabla \mathcal{G} \nabla \cdot \mathbf{m},$$

where the Green's-function operator \mathcal{G} is defined, upon operation on a regular function $f(\mathbf{r})$, as

$$\mathcal{G}f(\mathbf{r}) = \int_{\text{all space}} d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} f(\mathbf{r}').$$

We make the following definitions:

$$\partial^{\pm} \equiv \partial/\partial x \pm i\partial/\partial y,$$

$$h_0 \equiv (\mathbf{h}_s)_x + i(\mathbf{h}_s)_y, \quad \alpha \equiv m_x + im_y.$$

Equation (1) can then be rewritten as

$$\begin{aligned} \dot{\alpha} = & i[(H_0 + Gz)\alpha \\ & - (1 - \frac{1}{2}|\alpha|^2)(\frac{1}{2}\partial^+ \mathcal{G} \partial^- \alpha + \frac{1}{2}\partial^+ \mathcal{G} \partial^+ \alpha^* + h_0) \\ & - \Lambda \nabla^2 \alpha + \lambda \dot{\alpha}]. \end{aligned} \quad (2)$$

Assume Eq. (2) possesses a solution of the form

$$\alpha(\mathbf{r}, t) = \alpha(z, t) \sin(\pi x/a) \sin(\pi y/b), \quad (3)$$

with a and b being the lateral cross dimensions of the sample. Equation (3) corresponds to solutions which minimize the exchange energy, and, under uniform rf excitations, Eq. (3) represents the closest approximation of the real solution possessing definite k_x and k_y values satisfying boundary conditions at the sample boundaries. Nevertheless, it can never satisfy Eq. (2) rigorously, since the nonlinearity in Eq. (2) will couple the above solutions to other higher-order modes of k_x and k_y values. However, since the nonlinearity is small, we may assume Eq. (3) satisfies Eq. (2) in the sense of an average, i.e., Eq. (3) is substituted in Eq. (2) which is then followed by taking the average over the x and y dimensions. This is the only approximation that we will make in this theoretical treatment. This approximation is roughly valid if the excitation power of the signals is not too high. Under high-power excitations, energy will be significantly transferred from the lowest excitation state, Eq. (3), into higher-order modes, and Eq. (3) can still be adequately used if the damping constant λ in Eq. (2) is modified by including a power-dependent term:

$$\lambda \rightarrow [1 + O(|\alpha|^2)]\lambda.$$

This implies that nonlinear terms involving λ and Λ originally ignored in Eq. (1) shall be retained in Eq. (2).

Equation (2) can then be written as

$$\begin{aligned} \dot{\alpha} = & i \left[(H_0 + \Lambda p^2)\alpha - \Lambda \frac{\partial^2 \alpha}{\partial z^2} + \lambda \dot{\alpha} \right. \\ & \left. - \frac{\pi^2}{4} (1 - \frac{2}{9}|\alpha|^2)(h_1 + h_2) - \frac{\pi^2}{4} (1 - \frac{1}{8}|\alpha|^2)h_0 \right], \end{aligned} \quad (4a)$$

where h_1 and h_2 are defined as

$$h_1 \equiv -\frac{4p}{\pi} \int_{-\infty}^{\infty} dz' e^{-p|z-z'|} \alpha(z', t), \quad (4b)$$

$$h_2 \equiv -\frac{4q^2}{\pi p} \int_{-\infty}^{\infty} dz' e^{-p|z-z'|} \alpha(z', t)^*. \quad (4c)$$

The nonlocal effects are expressed in the above two fields, h_1 and h_2 . In the limit that p goes to infinity the

integrations in the above expressions are over narrow regions. Hence, h_1 and h_2 could only represent some sort of local field. p and q are defined as

$$p = [(\pi/a)^2 + (\pi/b)^2]^{1/2},$$

$$q = [(\pi/a)^2 - (\pi/b)^2]^{1/2}.$$

In deriving Eqs. (4a)-(4c) we have used the expression

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2\pi^2} \int_{\text{all space}} d\mathbf{k} \frac{e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}}{k^2}.$$

In the following we will omit the dipolar field h_2 by assuming $a \approx b$ and hence $q \approx 0$. This corresponds to the general circular-precession approximation imposed in the normal linear-mode calculations and is equivalent to neglecting the third Holstein-Primakoff transformation for ordinary spin waves.¹¹ Non-circular-precessional fields will have a profound effect on echo amplification if the sample possesses very different lateral cross dimensions.

Suppose the excitation field is of the form

$$h_0(t) = A_0 \delta(t) + \epsilon_0 \delta(t + \tau),$$

where A_0 and ϵ_0 ($\ll 1$) are the amplitudes of the pump and the signal pulses applied at $t=0$ and $t=-\tau$, respectively. Under the above excitation the solution of Eq. (4a) may be written in the form

$$\begin{aligned} \alpha(z, t) = & \Psi(t) [A e^{iGzt} + \epsilon \psi_1(t) e^{iGz(t+\tau)} \\ & + \epsilon \psi_2(t) e^{iGz(t-\tau)}], \end{aligned} \quad (5)$$

such that Eq. (4a) becomes separable in the variables z and t at least up to first order in ϵ . Integrating Eq. (4a) from $t=0^-$ to $t=0^+$, and from $t=-\tau^-$ to $t=-\tau^+$ the integral-differential equation can be treated as an initial-value problem satisfying, for $t > 0$, the following equation:

$$\begin{aligned} \dot{\alpha} = & i \left\{ [(1 + i\lambda)(H_0 + \Lambda p^2) + Gz]\alpha - \Lambda \frac{\partial^2 \alpha}{\partial z^2} \right. \\ & \left. - \frac{\pi^2}{4} h_1 (1 + i\lambda - \frac{2}{9}|\alpha|^2) \right\}, \end{aligned} \quad (6)$$

with the initial values

$$\Psi(0) = 1, \quad \psi_1(0) = 1, \quad \psi_2(0) = 0.$$

In the above we have set the phases of the initial values arbitrarily to zero. The two parameters A and ϵ are related to the excitation amplitudes A_0 and ϵ_0 in the following way:

$$|A_0| = \frac{4}{\pi^2} \frac{A}{1 - A^2/32}, \quad (7a)$$

$$|\epsilon_0| = \frac{4}{\pi^2} \epsilon, \quad (7b)$$

and we have assumed $\lambda \ll 1$ in writing Eq. (6). The

detected output is the averaged value of h_1 :

$$\begin{aligned} \langle h_1 \rangle &= \lim_{L \rightarrow \infty} \frac{-4p}{\pi} \int_{-L}^L \frac{dz}{2L} \int_{-\infty}^{\infty} e^{-p|z-z'|} \alpha(z') dz' \\ &= (-8/\pi) \langle \alpha \rangle. \end{aligned} \quad (8)$$

The echo gain is, therefore,

$$\text{gain} = 2\pi |\Psi(\tau) \psi_2(\tau)|. \quad (9)$$

The dipolar field associated with expression (5) can be written as

$$\begin{aligned} h_1(z, t) &= \frac{-8p}{\pi} \Psi(t) \left[A \frac{e^{iGzt}}{p - iGt} + \epsilon \psi_1(t) \frac{e^{iGz(t+\tau)}}{p - iG(t+\tau)} \right. \\ &\quad \left. + \epsilon \psi_2(t) \frac{e^{iGz(t-\tau)}}{p - iG(t-\tau)} \right]. \end{aligned} \quad (10)$$

Before proceeding further we notice here that, since only the averaged value of α matters in the final answer, Eqs. (8) and (9), the linear terms in h_1 in Eq. (6) may be replaced by $(-8/\pi)\alpha$ to simplify the following calculations. Furthermore, we may translate the origin of the z coordinate such as to cancel the linear terms in α possessing pure imaginary coefficients. The effective dc field is, therefore,

$$H_{dc} = H_0 + \Lambda p^2 + 2\pi,$$

where the last component 2π can be visualized as the rf demagnetization associated with the transverse directions of the sample ($N_x = N_y = \frac{1}{2}$). We introduce the parameter ν as

$$\nu = H_{dc} \lambda.$$

Note that H_{dc} determines the location of the active region of the sample and hence the carrier frequency of the excitation pulses. After expressions (5) and (10) have been substituted in Eq. (6), one may set the coefficients corresponding to $\exp(iGzt)$, $\exp[iGz(t+\tau)]$, and $\exp[iGz(t-\tau)]$ separately to zero and, after some mathematical manipulations, one obtains

$$\dot{W} = \frac{-\Gamma g t e^{-2\nu t}}{1 + g^2 t^2} W^3, \quad (11a)$$

$$\dot{\phi}_1 = i \left\{ \left[B - \frac{\Gamma W^2 e^{-2\nu t}}{1 - i g(t+1)} \right] \phi_1 - \frac{G W^2 e^{-2\nu t}}{1 - i g t} \phi_2^* \right\}, \quad (11b)$$

$$\dot{\phi}_2 = i \left\{ \left[B - \frac{\Gamma W^2 e^{-2\nu t}}{1 - i g(t-1)} \right] \phi_2 - \frac{G W^2 e^{-2\nu t}}{1 - i g t} \phi_1^* \right\}, \quad (11c)$$

where

$$W(t) \equiv |\Psi(t)| \exp(\nu t),$$

$$\phi_1(t) \equiv \psi_1(t) \exp(i\Lambda G^2 \tau t^2),$$

$$\phi_2(t) \equiv \psi_2(t) \exp(i\Lambda G^2 \tau t^2),$$

satisfying the initial values

$$W(0) = 1, \quad \phi_1(0) = 1, \quad \phi_2(0) = 0.$$

The echo gain may be expressed as

$$\text{gain} = 2\pi W(1) |\phi_2(1)| \exp(-\nu \tau).$$

Note that in Eqs. (11a)–(11c) the time variable has been normalized with respect to the pulse delay time τ and the parameters B , g , and Γ are defined as

$$B \equiv \Lambda G^2 \tau^3, \quad g \equiv G\tau/p, \quad \Gamma \equiv (4\pi/9) A^2 \tau.$$

Under lossless ($\nu=0$) and local ($g=0$) approximations Eqs. (11a)–(11c) can be solved analytically via Laplace transformation as

$$W(t) = 1, \quad (12a)$$

$$\phi_1(t) = \cosh \beta t + [(B - \Gamma)/\beta] \sinh \beta t, \quad (12b)$$

$$\phi_2(t) = (-i\Gamma/\beta) \sinh \beta t, \quad (12c)$$

where

$$\beta \equiv [B(2\Gamma - B)]^{1/2}.$$

Solution (12a)–(12c) is identical to that calculated by Herrmann, Hill, and Kaplan⁸ utilizing a hypothetical equation of motion. The unknown parameter q used by Herrmann, Hill, and Kaplan⁸ characterizing the local cubic nonlinearity is therefore identified in this paper as $q = 4\pi/9$. We must emphasize here that solution (12a)–(12c) may represent the real solution only when the delay time τ is small such that $g \ll 1$. The local solution (12a)–(12c) characterizes a gain which increases exponentially with the delay time. It is the nonlocality g which must be introduced in the equation of motion in order to bring the gain back to zero at large values of τ .

Let us now calculate the gain for realistic echo conditions. Equations (11a)–(11c) can be solved numerically using the fourth-order Runge-Kutta method. For 100 steps this method provides solutions with errors less than one part in ten thousand as corresponding solutions are compared with their analytic counterparts, Eqs. (12a)–(12c). Figure 1 shows a typical gain characteristic admitted by Eqs. (11a)–(11c), where we have used

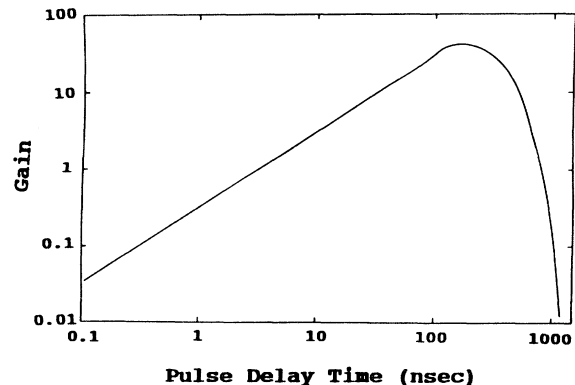
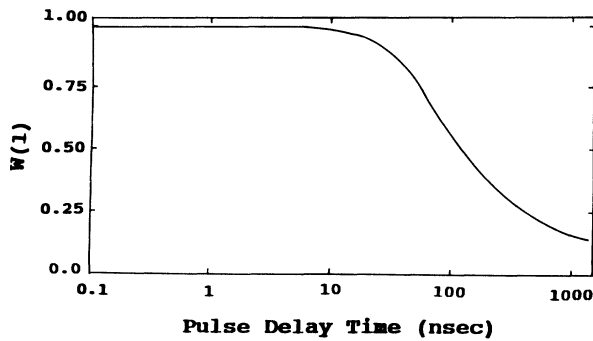


FIG. 1. Echo gain vs delay time.

FIG. 2. Plot of $W(1)$ as a function of τ .

the following parameters:

$$\gamma H_{dc} = 1 \text{ GHz}, \quad 4\pi M_s = 1750 \text{ Oe},$$

$$\Lambda = 4.13 \times 10^{-11} \text{ cm}^2, \quad \lambda = 5.4 \times 10^3 \text{ sec}^{-1},$$

$$a = b = 1 \text{ mm}, \quad G = 1000 \text{ Oe/cm}, \quad A_0 = 16.8 \text{ Oe}.$$

The cw power supplied by the pump signal is calculated to be 1.1 W assuming the carrier frequency of 1 GHz and an active sample region of volume 100 mm^3 . We must emphasize here that Fig. 1 resembles exactly the experimental curves in Ref. 2 in both magnitude and qualitative behavior with respect to τ . Figure 2 shows a plot of $W(1)$ as a function of delay time τ . When multiplied by $\exp(-\nu\tau)$, Fig. 2 represents the amplitude of the magnetic moment excited by the pump pulse at the instant that the echo signal occurs. Under local and lossless approximations $W(1)$ assumes the constant value of unity as indicated by Eq. (12a). It is the nonlocality g that reduces $W(1)$ to zero at large τ values.¹²

We summarize this paper by the following. Based upon the order of approximation of previous work we have added extra terms in the equation of motion to account for magnetic relaxation and long-range dipole-dipole interaction effects on the excitation of echoes.

These effects were purposely introduced by us to determine the reduction of the rf-magnetic-moment amplitude for times greater than the excitation pulse width. The effect of the long-range dipole-dipole interaction is to reduce the internal rf magnetic field to zero and hence the rf magnetic moment. This means that although the rf magnetic moments are in phase upon the application of the excitation pulse the dipole-dipole interaction will tend to uncorrelate the rf moments and eventually reduce the moment to zero after sufficient elapsed time. Hence, the moment reduces to zero even if the relaxation time is infinite. Of course, with finite relaxation times as we have assumed in our calculation it can only increase the rate of rf-magnetic-moment reduction as shown in Fig. 2. Once the moment is reduced to zero, the gain will be small. Our calculations demonstrate this effect; see Figs. 1 and 2.

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¹² $W(1)$ can be solved analytically in the nonlocal and lossless case as $W(1) = [1 + (\Gamma/g)\ln(1+g^2)]^{-1/2}$, which reduces to zero logarithmically for large values of τ .