Guiding-Center Soliton

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We consider the behavior of solitons described by the nonlinear Schrödinger equation with periodic perturbation (not necessarily small) whose period Z_a is much shorter than the characteristic period of the solitons, Z_0. The soliton behaves like the guiding-center motion of a charged particle in a magnetic field, smooth and adiabatic. However, when Z_a approaches Z_0, resonances appear between the periodic perturbations and the characteristic frequencies of the solitons which induce the generation of dispersive waves and/or the splitting of solitons.

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We consider the behavior of solitons described by the nonlinear Schrödinger equation with periodic perturbation,

$$\frac{\partial q}{\partial Z}=i\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+i|q|^2q+f(Z)q\,, \tag{1}$$

or equivalently by

$$\frac{\partial u}{\partial Z}=i\frac{1}{2}\frac{\partial^2 u}{\partial T^2}+ia^2(Z)|u|^2u\,, \tag{2}$$

where

$$q(Z,T)=a(Z)u(Z,T),\ \text{with}\ \frac{da}{dZ}=f(Z)a\,. \tag{3}$$

Here Z is the distance of propagation normalized to the dispersion distance, $f(Z)$ is a periodic function in Z with the period Z_a *much smaller* than the dispersion distance $(=1)$.

In this Letter, we show that, based on the Lie transformation and the averaging method, the nonlinear Schrödinger solitons are remarkably robust in spite of such rapid perturbations with a relatively large amplitude.

One important example is the propagation of optical solitons in a dielectric fiber in which the solitons are periodically amplified to the initial amplitude, where $f(Z)$ is given by[1]

$$f(Z)q=-\Gamma q+(e^{\Gamma Z_a}-1)\sum_{n=1}^{N}\delta(Z-nZ_a)q(nZ_a-0,T)\,. \tag{4}$$

Here Γ is the exponential loss rate *per dispersion distance* of the fiber in the Z direction and $e^{\Gamma Z_a}-1$ is the gain of the amplifier placed at $Z=nZ_a$ to compensate for the fiber loss between two amplifiers. If $\Gamma\ll 1$, solitons are known to behave adiabatically with a slow variation in their parameters.[2] The case we consider $(Z_a\ll 1)$ has a rapidly varying perturbation, which allows $\Gamma\gg 1$, where the adiabatic perturbation theory[3] based on the smoothness of the perturbation is not applicable.

The situation is analogous to the motion of a charged particle in a slowly varying electromagnetic field where the exact position oscillates rapidly due to the Larmor motion, yet the center of the Larmor motion (the guiding center) moves smoothly on a distance scale much larger than the Larmor radius. Consequently, we solve the behavior of q in Eq. (1) by transforming u to a guiding-center variable v which is integrable to $O(Z_a^2)$ by means of the Lie transformation and averaging. The quantity v satisfies the nonlinear Schrödinger equation to $O(Z_a^2)$ except for a case where resonance occurs between the periodicity of $v(Z,\cdot)$ and of $f(Z)$.

The first part of the Letter describes the derivation of the transformed equation for $v(Z,T)$ for the case of nonresonance $(Z_a\ll 1)$, and the second part, the effects of the resonance on soliton propagation when Z_a approaches unity.

By analogy to the derivation of the guiding-center motion,[4] we introduce the Lie transformation which is extended to a variable with an infinite degree of freedom,[5,6]

$$u=e^{\boldsymbol{\phi}\cdot\nabla}v=v+\phi(v,v^*,Z)+\tfrac{1}{2}(\boldsymbol{\phi}\cdot\nabla\phi)(v,v^*,Z)+\cdots\,, \tag{5}$$

where $\boldsymbol{\phi}=(\phi,\phi^*)$ and the directional derivative $\boldsymbol{\phi}\cdot\nabla$ is defined as

$$\boldsymbol{\phi}\cdot\nabla=\sum_{n=0}^{\infty}\left(\phi_{nT}\frac{\partial}{\partial v_{nT}}+\phi_{nT}^*\frac{\partial}{\partial v_{nT}^*}\right)\,, \tag{6}$$

with $\phi_{nT}=\partial^n\phi/\partial T^n$ and $v_{nT}=\partial^n v/\partial T^n$. The transformed quantity v is expressed in terms of a variable of infinite dimension $(v,v^*,v_T,v_T^*,v_{2T},v_{2T}^*,\dots)$.

The evolution equation of u, Eq. (2), is expressed in the form

$$\frac{du}{dZ}=X(u,u^*;Z)=X_0(u,u^*)+\tilde{A}(Z)\bar{X}_0(u,u^*)\,, \tag{7}$$

with

$$X_0(u,u^*)=\frac{i}{2}\frac{\partial^2 u}{\partial T^2}+iA_0|u|^2u\,,\quad \bar{X}_0(u,u^*)=i|u|^2u\,.$$

Here, $A_0=\langle a^2(Z)\rangle$ is the average of $a^2(Z)$ over one period Z_a, $\tilde{A}(Z)=a^2(Z)-A_0$ is a periodic function with periodicity Z_a whose average is zero, and

$$\frac{d}{dZ}=\frac{\partial}{\partial Z}+\frac{d\mathbf{v}}{dZ}\cdot\boldsymbol{\nabla}. \tag{8}$$

The averaged nonlinear Schrödinger equation is obtained in the form

$$\frac{dv}{dZ}=Y_0(v,v^*)=X_0(v,v^*)+Y_{01}(v,v^*)+\cdots, \tag{9}$$

where $Y_{0n}(v,v^*)=O(Z_a^n)$ is obtained successively by means of the Lie transformation, Eq. (5).

Inserting Eq. (5) into Eq. (7), we have

$$\begin{aligned}\frac{du}{dZ}&=\frac{dv}{dZ}+\frac{d\mathbf{v}}{dZ}\cdot\boldsymbol{\nabla}(\phi+\tfrac{1}{2}\boldsymbol{\phi}\cdot\boldsymbol{\nabla}\phi+\cdots)+\frac{\partial}{\partial Z}(\phi+\tfrac{1}{2}\boldsymbol{\phi}\cdot\boldsymbol{\nabla}\phi+\cdots)\\&=X(e^{\boldsymbol{\phi}\cdot\boldsymbol{\nabla}}v,e^{\boldsymbol{\phi}\cdot\boldsymbol{\nabla}}v^*;Z)=e^{\boldsymbol{\phi}\cdot\boldsymbol{\nabla}}X(v,v^*;Z).\end{aligned} \tag{10}$$

We now expand ϕ in the form

$$\phi=\phi_1+\phi_2+\cdots, \tag{11}$$

where ϕ_n is shown to be of $O(Z_a^n)$. In this expansion we note that $\partial\phi_n/\partial Z=O(Z_a^{n-1})$ but $dv/dZ=O(1)$, provided that there exists no resonance between the soliton solution of v and the periodic perturbation given by $f(Z)$. We will discuss the effects of this resonance later. We obtain the following equations for each order in the expansion of (10).

For order $Z_a^0=1$, we have

$$\phi_1(v,v^*,Z)=\tilde{A}_1(Z)\bar{X}_0(v,v^*), \tag{12}$$

where $\tilde{A}_1(Z)$ is the solution of

$$\frac{d\tilde{A}_1}{dZ}=\tilde{A}\ \text{with}\ \langle\tilde{A}_1\rangle=0, \tag{13}$$

which is the nonsecular condition for the expansion (11). The solution $\tilde{A}_1$ is given by

$$\tilde{A}_1(Z)=\int_0^Z\tilde{A}(Z')dZ'-\left\langle\int_0^Z\tilde{A}(Z')dZ'\right\rangle, \tag{14}$$

and thus we note, from $A(Z)>0$,

$$|\tilde{A}_1(Z)|\le 2A_0Z_a. \tag{15}$$

From $O(Z_a)$, we have

$$\frac{\partial\phi_2}{\partial Z}=\tilde{A}_1[\bar{X}_0,X_0]-Y_{01}, \tag{16}$$

where $[\bar{X}_0,X_0]=\bar{\mathbf{X}}_0\cdot\boldsymbol{\nabla}X_0-\mathbf{X}_0\cdot\boldsymbol{\nabla}\bar{X}_0$ is the Lie bracket which becomes

$$[\bar{X}_0,X_0]=-(2v|v_T|^2+v^*v_T^2+v^2v_{TT}^*). \tag{17}$$

In Eq. (16), since $\langle\tilde{A}_1\rangle=0$, $Y_{01}=0$ (the nonsecular condition). Then the solution of Eq. (16) is given by

$$\phi_2(v,v^*,Z)=\tilde{A}_2(Z)[\bar{X}_0,X_0], \tag{18}$$

where $\tilde{A}_2(Z)$ is the solution of

$$\frac{d\tilde{A}_2}{dZ}=\tilde{A}_1\ \text{with}\ \langle\tilde{A}_2\rangle=0. \tag{19}$$

From Eq. (15), we see that $\tilde{A}_2=O(Z_a^2)$.

Going to $O(Z_a^3)$ in a similar manner, we obtain the average nonlinear Schrödinger equation up to $O(Z_a^2)$,

$$\frac{\partial v}{\partial Z}=i\frac{1}{2}\frac{\partial^2 v}{\partial T^2}+iA_0|v|^2v+\tfrac{1}{2}\langle\tilde{A}\tilde{A}_2\rangle[\bar{X}_0,[\bar{X}_0,X_0]]+O(Z_a^3), \tag{20}$$

while

$$u=v+i\tilde{A}_1|v|^2v-\tfrac{1}{2}\tilde{A}_1^2|v|^4v-\tilde{A}_2(2v|v_T|^2+v^*v_T^2+v^2v_{TT}^*)+O(Z_a^3). \tag{21}$$

We note that A_0 can be made equal to unity by a proper choice of the integral constant $a(Z)$ or by a proper scaling. Thus, the transformed variable v satisfies the canonical nonlinear Schrödinger equation to $O(Z_a^2)$, and hence the guiding-center soliton solution described by Eq. (20) is valid to a distance $O(Z_a^{-2})$. This fact was verified by numerical simulation of Eq. (1) with $f(Z)$ given by Eq. (4).[7]

We now discuss the effect of resonances. When Z_a approaches Z_0, the nth harmonic of the periodicity Z_a, $2\pi n/Z_a$, can have resonances with characteristic nonlinear oscillations of the guiding-center soliton solutions of Eq. (20) and the adiabaticity is expected to be broken. Here we numerically consider two types of such resonances. One is the resonance of the one-soliton solution and the other is that of the two-soliton solution.

The one-soliton resonance condition is given by

$$\eta^2/2=2\pi n/Z_a,\quad n=1,2,\ldots, \tag{22}$$

where η is the soliton amplitude and $Z_0=4\pi/\eta^2$ is the one-soliton oscillation period. We studied the effect of the $n=1$ resonance by solving Eq. (1) with Eq. (4) for the choice of $Z_a=0.5$, $\Gamma=0.23$, and $a(Z)=a_0e^{-2\Gamma Z}$ with $a_0=[2\Gamma Z_a/(1-e^{-2\Gamma Z_a})]^{1/2}=1.06$, for which $A_0=1$ and $|\tilde{A}_1|\le 9.6\times10^{-3}$. A small Γ is chosen here to reduce the width of the resonance. The resonant soliton amplitude is given by $\eta=5.01$. Figure 1 shows the behavior of $|q(T)|$ at $Z=0,2,4,\ldots,12$ with the initial condition $q(0,T)=a_0\eta\,\mathrm{sech}\,\eta T$.

Note that dispersive waves are emitted periodically until the pulse amplitude decays to approximately 4. We

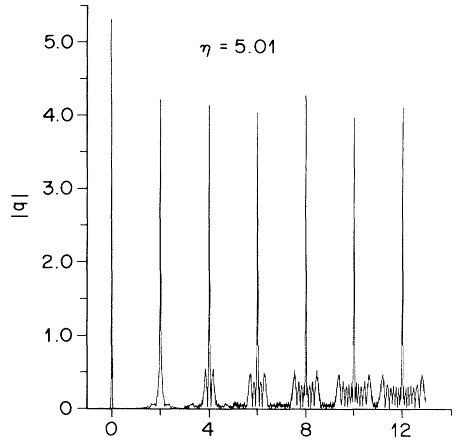

FIG. 1. Effect of one-soliton resonance $|q(T)|$ plotted at distance Z. At each distance of $Z=0,2,\ldots,12$, $|q(T)|$ is shown for $-25 \leq T \leq 25$. Dispersive waves are periodically emitted away from the soliton until the amplitude is reduced to the stable range.

ran similar calculations for various values of the initial amplitudes and found that if $\eta \gtrsim 5.5$ or $\eta \lesssim 4$, the emission of the dispersive waves diminished. We conclude that the one-soliton resonance enhances emission of dispersive waves until the soliton amplitude reaches a level which is outside of the resonance width determined by Z_a and Γ.

The resonance condition for two bound solitons with η_1 and η_2 is given by

$$\tfrac{1}{2}(\eta_1^2-\eta_2^2)m=(2\pi/Z_a)n, \quad m,n=1,2,\ldots, \tag{23}$$

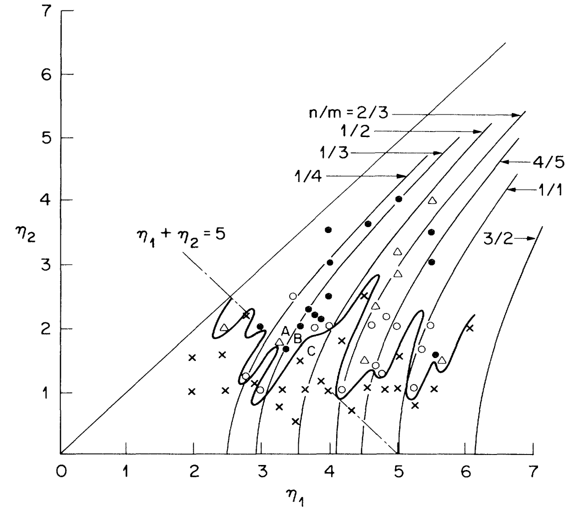

FIG. 2. Stable (crosses) and unstable (circles) regions in the initial amplitudes η_1 and η_2 space for two bound solitons. Instability is identified by the separation into two solitons with identical amplitudes $\simeq(\eta_1+\eta_2)/2$. Open and closed circles are slow and abrupt separations, while triangles are the case of no clear sign.

where Z_0 in this case is $4\pi/(\eta_1^2-\eta_2^2)$, the two-soliton period.[8] For a given value of Z_a ($=0.5$), the resonant condition is given by a set of curves in the η_1-η_2 plane for each n/m ratio.

A large number of numerical calculations of Eq. (1) with Eq. (4) were performed for various sets of initial eigenvalues η_1 and η_2 for the initial value of q given by the bound-two-soliton shape,[9]

$$q(0,T)=\frac{2a_0}{\Delta}\left(\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right)(\eta_1\cosh\eta_2 T e^{i\theta_1}+\eta_2\cosh\eta_1 T e^{i\theta_2}),$$
$$\Delta=\left(\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right)^2\cosh(\eta_1+\eta_2)T+\cosh(\eta_1-\eta_2)T+\frac{4\eta_1\eta_2}{(\eta_1+\eta_2)^2}\cos(\theta_1-\theta_2). \tag{24}$$

In the absence of the perturbation, the phase θ_i evolves according to

$$\theta_i=-\tfrac{1}{2}\eta_i^2 Z+\theta_{i0}. \tag{25}$$

It was found that the initially bound solitons given by Eq. (24) separated under resonant conditions and two solitons with identical amplitudes (as is expected from the conservation law) emerged. This indicates that the two-soliton resonances excited by the periodic perturbation induce merging of the eigenvalues η_1 and η_2.

Figure 2 gives the results of the numerical calculations ($Z \lesssim 20$) where cases of abrupt separation (solid circles), slow separation (open circles), and no separation (crosses) are shown in the plane of initial values of η_1 and η_2. Triangles are the cases which are not clearly discernible. An abrupt separation is identified as one which took place within a distance of $Z<10$, while slow separations are those of $Z \gtrsim 10$. The curves show resonance lines of Eq. (23) for $n/m=1/4$, etc., as indicated. There exists an infinite set of n and m combinations which satisfy the resonances, but only lower integer values are shown. The solid curve shows the demarcation line between the unstable (separation) and stable (nonseparation) regions. There are indications of resonant effects in that the demarcation line parallels the resonant curves.

We note from Eq. (2) that $|u|^2$ is conserved; thus if we ignore radiation,

$$I_1=\tfrac{1}{2}\int|u|^2 dT=\eta_1+\eta_2=\text{const},$$

while the quantity which is conserved for the unperturbed case,

$$I_2=\tfrac{3}{2}\int(|u|^4-|u_T|^2)dT=\eta_1^3+\eta_2^3$$

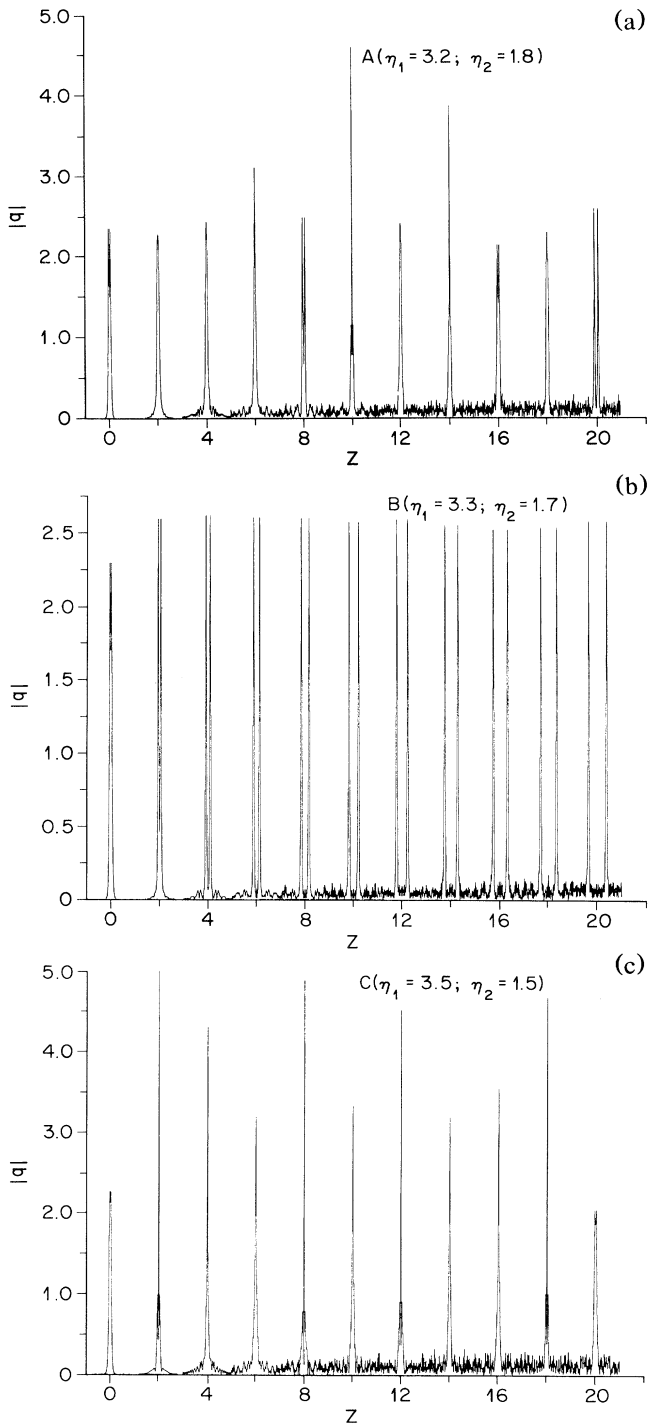

FIG. 3. Variation of bound-two-soliton magnitude $|q(T)|$ at distances $Z=0,2,\ldots,20$ with initial eigenvalues η_1 and η_2 being (a) (3.2,1.8), (b) (3.3,1.7), and (c) (3.5,1.5). Note the abrupt splitting into two solitons in (b) while there is stable propagation in (c) and marginally stable propagation in (a).

is not conserved. Hence in order for a resonance to break a soliton, it may require a process which leads to a reduction of I_2 such that $\eta_1=\eta_2=I_1/2=\eta$ is achieved. This process will be discussed in a future publication.[6]

Figure 3 shows the behavior $|q(T)|$ at $Z=0,2,4,\ldots,20$ of two bound solitons for the initial amplitudes of (a) ($\eta_1=3.2$, $\eta_2=1.8$), (b) ($\eta_1=3.3$, $\eta_2=1.7$), and (c) ($\eta_1=3.5$, $\eta_2=1.5$) all lying on $\eta_1+\eta_2=5$. Note that case (b) produces abrupt separation, while case (c) is stable and case (a) is marginally stable. Note also that both stable (nonseparation) and unstable (separation) points exist on the line $\eta_1+\eta_2=5$. This indicates that there exists no direct dynamical connection of phase space along the line of $I_1=$const. For example, the instability of case (b) ($\eta_1=3.3$, $\eta_2=1.7$), which has rapidly changed the eigenvalues η_1 and η_2 to 2.5, did not take place by moving through the stable point near $\eta_1=3.2$ and $\eta_2=1.8$, an indication of Arnold-type diffusion. This is a consequence of the fact that the system has infinite degrees of freedom.

In conclusion, by means of the Lie transformation and successive averaging we have shown that the soliton (the guiding-center soliton) described by the nonlinear Schrödinger equation with rapidly varying perturbations is quite robust even if the period of the perturbation is much shorter than the characteristic period of a soliton solution of the unperturbed equation. When a resonance occurs between these periods, the guiding-center soliton may split into two (or more) solitons or emit continuous waves. The result is immediately applicable to the long-distance propagation of optical solitons which are repeatedly amplified. Furthermore, the present method of Lie transform and averaging may be used to show the robust properties of other soliton systems in the presence of rapidly varying periodic perturbations.

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