Hermitian Structure for the Linearized Vlasov-Poisson and Vlasov-Maxwell Equations

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The *linearized* Vlasov-Poisson and Vlasov-Maxwell equations are shown to have a structure closely related to the evolution equation of quantum mechanics, in terms of a nonstandard Hilbert space. This Hermitian structure yields information about spectral properties, as well as a theory for dynamical invariants. We find accordingly how certain well-known features of the spectrum generalize to the nonuniform case, and we also rederive a recently found exact dynamical invariant in a very natural and simple way.

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The analysis of linearized Vlasov-Poisson equations by van Kampen and Case some thirty years ago explicitly determined the spectrum and eigenfunctions in the unmagnetized uniform case.[1,2] These results have been generalized in several ways (see Sedlacek,[3] and references therein). However, van Kampen and Case found a non-Hermitian structure of the linearized Vlasov-Poisson operator, and this has been an obstacle in all these later developments. A Hermitian structure would, of course, have saved us from separate analysis of adjoint equations. Furthermore, when an operator is known to be self-adjoint, there is a body of general results and techniques promoting further developments.

In the present paper we find an underlying *Hermitian* structure of the *linearized Vlasov-Poisson* and *Vlasov-Maxwell* equations (hereafter referred to as the LVP and the LVM equations). The results are obtained for a *stationary* but otherwise *general nonuniform* background state. There are two major novel features in this theory. First, the space of fields is not given the structure of a *standard* Hilbert space. Instead we find a natural inner product which induces a *pseudometric* (i.e., an *indefinite* metric). We will use the term *pseudo-Hilbert space* for our spaces of fields equipped with such an inner product. Second, in place of the perturbed distribution function f as one of the fields, we use the generator S of particle-orbit perturbation. (There is, of course, a simple relationship between f and S.) In this way the LVP and LVM equations have a place within the well-developed general theory of Hermitian operators on spaces with indefinite inner products.[4] We obtain a theory in close analogy to quantum mechanics, and expect that future developments and applications of the theory will include the use of techniques from that area.

While the LVP and LVM equation provide the foundation for much analytical and numerical work in plasma physics, only very few results about their underlying mathematical structure have been known. The full *nonlinear* Vlasov-Poisson and Vlasov-Maxwell equations are Hamiltonian in terms of Poisson-manifold structures.[5–7] The LVP equations are known to inherit this property[8] and the LVM equations are expected to do so.[9] Recently, Morrison and Pfirsch[10] proved that a quantity interpreted as "the energy of perturbation" is an exact invariant of the LVM equations. (This result will be obtained in a very natural and simple way from our formalism.) These previous results are here complemented with a sound setting for the spectral theory of the linear operators associated with the LVP and LVM equations. In the case of a uniform unmagnetized background plasma the spectral properties of the LVP are known from the analysis of van Kampen and Case. Some further information was obtained by the technique of spectral deformation, borrowed from quantum mechanics and modified by Crawford and Hislop to meet the needs of plasma physics.[11] (This is also a good reference when we consider the consistency between these spectral results and the general mathematical structure we present.) The spectrum obtained when the plasma is *unstable* consists of a continuous straight line through the origin as well as isolated eigenvalues appearing away from this line and associated with unstable solutions of the usual dispersion relation. It is clear that an "essentially" Hermitian[12] operator on a *standard* Hilbert space cannot be associated with this spectrum. However, it *is* consistent with a Hermitian operator on a *pseudo-Hilbert* space.

We shall now show that both the LVP and the LVM equations are closely associated with Hermitian operators on pseudo-Hilbert spaces. There are certain nontrivial considerations needed in order to give the precise mathematical structure required for a good spectral theory to exist. As in most textbooks in quantum mechanics, we consider only the finite-dimensional case, assuming that the generalizations needed in reality are straightforward. This approach turns out to be instructive; it explains nicely from general theory some characteristic features of the spectrum obtained by van Kampen and Case, and shows that they persist also for a nonuniform plasma. For notational convenience we consider a one-component plasma and take $q=m=\varepsilon_0=\mu_0=1$. The generalization to several components is trivial.

The *stationary* background plasma is given by (F_0,

$\mathbf{E}_0,\mathbf{B}_0)$, where $F_0(\mathbf{r},\mathbf{v})$ denotes distribution function and $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$ electric and magnetic fields. The equations for the background imply

$$\{F_0,H_0\}=0,\quad \mathbf{E}_0=-\nabla\phi_0,\quad \mathbf{B}_0=\nabla\times\mathbf{A}_0. \tag{1}$$

Here $H_0(\mathbf{r},\mathbf{v})=\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+\phi_0(\mathbf{r})$, and the Poisson bracket is defined for arbitrary functions $a(\mathbf{r},\mathbf{v})$ and $b(\mathbf{r},\mathbf{v})$ by

$$\{a,b\}\equiv\partial_{\mathbf{r}}a\cdot\partial_{\mathbf{v}}b-\partial_{\mathbf{r}}b\cdot\partial_{\mathbf{v}}a+\mathbf{B}_0\cdot\partial_{\mathbf{v}}a\times\partial_{\mathbf{v}}b.$$

Equations (1) give the Vlasov equation for the background distribution, as well as $\nabla\times\mathbf{E}_0=0$ and $\nabla\cdot\mathbf{B}_0=0$.

Both the LVP and the LVM equations are closely related to equations with the following general mathematical structure. There is a Hermitian operator $\hat{H}$ defined on a pseudo-Hilbert space $\mathcal{F}$ with inner product $\langle\,,\rangle$. The dynamical equation for $\psi\in\mathcal{F}$ is

$$\hat{H}\psi=i\partial_t\psi. \tag{2}$$

In the LVP case we take $\mathcal{F}$ as a space of time-dependent complex-valued fields S on $(\mathbf{r},\mathbf{v})$ space. For S_1 and S_2 in $\mathcal{F}$, we define their inner product by[13]

$$\langle S_1,S_2\rangle\equiv i\int\{S_1^*,S_2\}F_0\,d^3r\,d^3v. \tag{3}$$

Note that $\langle S,S\rangle=0$ for real-valued S. We define the operator $\hat{H}$ by

$$\hat{H}S\equiv-i\{S,H_0\}+i\phi(S), \tag{4a}$$

where the scalar potential $\phi(S)$ depends linearly on S as

$$\phi(S)(t,\mathbf{r})\equiv-\int\frac{\{F_0,S\}(t,\mathbf{r}',\mathbf{v}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r'\,d^3v'. \tag{4b}$$

We can now prove that $\hat{H}$ is Hermitian, i.e., $\langle S_1,\hat{H}S_2\rangle=\langle\hat{H}S_1,S_2\rangle$. We note that by defining $f\equiv-\{F_0,S\}$, the LVP equations are obtained from (2) and (4) [operate on (4a) with $\{\cdot,F_0\}$]. It is also easy to see that $\hat{H}$ has the same spectrum as $i\hat{L}$, where $\hat{L}$ is the usual linearized Vlasov-Poisson operator.[11] The proof that $\hat{H}$ is Hermitian is a straightforward calculation, using the rules for manipulating Poisson brackets and partial integrations.

In the LVM case (using the radiation gauge) we have $\mathcal{F}$ as a space of time-dependent complex-valued fields $\psi=(S,\mathbf{E},\mathbf{A})$, where S is defined on $(\mathbf{r},\mathbf{v})$ space as before, while the electric- and vector-potential perturbations $\mathbf{E}$ and $\mathbf{A}$ are defined on $\mathbf{r}$ space. The inner product is defined as

$$\langle\psi_1,\psi_2\rangle\equiv i\int\{S_1^*,S_2\}F_0\,d^3r\,d^3v+i\int(\mathbf{E}_1^*\cdot\mathbf{A}_2-\mathbf{E}_2\cdot\mathbf{A}_1^*)d^3r, \tag{5}$$

and we define $\hat{H}$ by

$$\hat{H}\psi\equiv i(-\{S,H_0\}-\mathbf{v}\cdot\mathbf{A},\ \nabla\times(\nabla\times\mathbf{A})-\mathbf{J}(S,\mathbf{A}),\ -\mathbf{E}), \tag{6a}$$

where the perturbed current density is

$$\mathbf{J}(S,\mathbf{A})\equiv\int\mathbf{v}(-\{F_0,S\}+\mathbf{A}\cdot\partial_{\mathbf{v}}F_0)d^3v. \tag{6b}$$

We can then easily prove that $\hat{H}$ is Hermitian, i.e., $\langle\psi_1,\hat{H}\psi_2\rangle=\langle\hat{H}\psi_1,\psi_2\rangle$. We define $f\equiv-\{F_0,S\}+\mathbf{A}\cdot\partial_{\mathbf{v}}F_0$, and it is straightforward (but now with a little more algebra than in the LVP case) to obtain the LVM equations from (2) and (6).

We obtain some information about the difference between the spectral structure of a Hermitian operator on a Hilbert space and on a pseudo-Hilbert space by the following linear-algebra result (I) and its generalization (II).

(I) Let V be a finite-dimensional standard Hilbert space and let $\hat{H}$ be a Hermitian operator on V. Then there exists an orthonormal basis $(\psi_1,\ldots,\psi_n)$ for V, consisting of eigenvectors for $\hat{H}$; i.e., (a) $\hat{H}\psi_j=\lambda_j\psi_j$, (b) $\langle\psi_i,\psi_j\rangle=\delta_{ij}$.

(II) Let V be a finite-dimensional vector space V with an *indefinite* inner product which is *nondegenerate*[14] (i.e., if $\langle\psi',\psi\rangle=0$ for all $\psi'\in V$, then $\psi=0$). Let $\hat{H}$ be a Hermitian operator on V. Then there exists a basis B of V consisting of the eigenvectors $\psi_1,\ldots,\psi_n$ for $\hat{H}$ and a 1-1 mapping $B\rightarrow B$, which we denote by superscript "†" as $\psi\rightarrow\psi^\dagger$, such that (a) $\hat{H}\psi_j=\lambda_j\psi_j$; (b) if λ_j is real, then $\psi_j^\dagger=\psi_j$; (c) if $\psi_j^\dagger=\psi_i$, then $\lambda_i^*=\lambda_j$ where "$*$" means complex conjugation; (d) $|\langle\psi_i^\dagger,\psi_j\rangle|=\delta_{ij}$.

The proof of these two results consists of the (generalized) Gram-Schmidt orthogonalization procedure using the equality $(\lambda_1^*-\lambda_2)\langle\psi_1,\psi_2\rangle=0$, valid for eigenfunctions ψ_1 and ψ_2, such that $\hat{H}\psi_i=\lambda_i\psi_i$ for $i=1,2$.

Consider now an eigenvector ψ_1 with a *nonreal* eigenvalue λ_1, for the LVP or the LVM case. A slight extension of result (II) tells us that we are allowed to let ψ_1 be an element in the basis B. Then $\psi_2=\psi_1^\dagger$ is also an eigenvector with eigenvalue $\lambda_2=\lambda_1^*$. By complex conjugating $\hat{H}\psi_1=\lambda_1\psi_1$, we obtain $\hat{H}\psi_1^*=-\lambda_1^*\psi_1^*$ [from (4) or (6) we have $\hat{H}^*=-\hat{H}$]. Thus a third eigenvector $\psi_3=\psi_1^*$ with eigenvalue $\lambda_3=-\lambda_1^*$ is obtained. Finally, we have $\psi_4=\psi_3^\dagger$ with $\lambda_4=\lambda_3^*=-\lambda_1$. Thus, in the general case, nonreal points in the spectrum appear in quadruplets $(\lambda_1,\lambda_1^*,-\lambda_1,-\lambda_1^*)$. We also have $\langle\psi_i,\psi_j\rangle=0$ when $\lambda_i\neq\lambda_j^*$, which in the particular case considered by Case and van Kampen may be seen to follow explicitly from the dispersion relation. Precisely this quadruplet structure is seen in the Case–van Kampen analysis (e.g., Ref. 11, Fig. 1).

Let us now consider dynamical invariants associated with (II). If a time-independent operator $\hat{A}$ commutes with $\hat{H}$, then $\langle\psi,\hat{A}\psi\rangle$ is a dynamical invariant. For $\hat{A}=\hat{H}^n$ $(n=1,2,\ldots)$, we thus find an infinite set of dynamical invariants. We easily rederive the Morrison-Pfirsch result [Ref. 10, Eq. (13)] by calculating $\langle\psi,\hat{H}\psi\rangle$, using (5) and (6):

$$\langle\psi,\hat{H}\psi\rangle=\int(\{S^*,F_0\}\{H_0,S\}+\mathbf{A}^*\cdot\mathbf{A}F_0-\{S^*,F_0\}\mathbf{v}\cdot\mathbf{A}-\{S,F_0\}\mathbf{v}\cdot\mathbf{A}^*)d^3r\,d^3v+\int(\mathbf{E}\cdot\mathbf{E}^*+\mathbf{B}\cdot\mathbf{B}^*)d^3r.$$

If the background has a symmetry, we may find an associated dynamical invariant, such as the angular momentum of a tokamak, or the helical momentum in the wiggler field of a free-electron laser. Mathematically, we must find the operator associated with each symmetry. A time-independent vector field $\mathbf{g}$ on ordinary $\mathbf{r}$ space has an associated operator $\hat{\mathcal{L}}_{\mathbf{g}}$ acting on the fields as the appropriate Lie derivative. The background state is symmetric with respect to $\mathbf{g}$ if $\hat{H}$ and $\hat{\mathcal{L}}_{\mathbf{g}}$ commutes. Then $\langle \psi, \hat{\mathcal{L}}_{\mathbf{g}} \psi \rangle$ is the corresponding dynamical invariant.

The formulas and the mathematical structure for the LVP and LVM equations were found by using an action principle related to the well-known formulation by Low.[15] By noncanonical Hamiltonian analysis, we obtained the basic mathematical structure presented in this paper from the Euler-Lagrange equations.[16]

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[12]By an "essentially" Hermitian operator we mean an Hermitian operator multiplied with a complex number.

[13]A structure similar to (3) but for the full *nonlinear* Vlasov-Poisson equations appears as Eq. (18) in J. D. Crawford and P. D. Hislop, Phys. Lett. A **134**, 19 (1988). It is there referred to as the Kirillov-Kostant-Souriau form, and the Poisson structure on a symplectic leaf may be expressed in terms of it.

[14]Rather much attention has been paid to the fact that the LVP and LVM equations are associated with an infinite degeneracy at eigenvalue $\lambda = 0$. This degeneracy is represented by the kernel K of the inner product, $K = \{\psi | \langle \psi', \psi \rangle = 0$ for all $\psi' \in \mathcal{F}\}$. In order to have a nondegenerate inner product we replace $\mathcal{F}$ by $\mathcal{F}/K$.

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