Dual Spectra and Mixed Energy Cascade of Turbulence in the Wavelet Representation

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The wavelet-transformed Navier-Stokes equations are used to define quantities such as the transfer and flux of kinetic energy through position $\mathbf{x}$ and scale r. Analysis of pseudospectral direct numerical simulations of turbulent flows reveals that although the mean spatial values of these quantities agree with their traditional counterparts in Fourier space, their spatial variability is very large, exhibiting non-Gaussian statistics. The local flux of energy involving scales smaller than some r also exhibits large spatial intermittency and it is negative quite often, indicative of local inverse cascades.

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Much has been learned about the physics of turbulence by transforming the velocity and the Navier-Stokes equations to Fourier space. The velocity field $\mathbf{u}(\mathbf{x},t)$ is then represented as a linear combination of physically extended plane waves, characterizing the motion at different scales. For isotropic and homogeneous turbulence, the energetics of turbulence[1] is described by the three-dimensional energy spectrum $E(k,t)$ obeying

$$\frac{\partial E(k,t)}{\partial t}=T(k,t)-2\nu k^2E(k,t)\,, \tag{1}$$

where $T(k,t)$ is the net transfer of energy to wave numbers of magnitude k. $T(k,t)$ is defined as triple products of fluctuating velocities and thus embodies the closure problem resulting from the nonlinearity of the equations. For the statistically stationary case, the total spectral flux of energy through wave number k to all smaller scales is given by

$$\pi(k)=\int_{k'=k}^{\infty}T(k')dk'=-\int_0^kT(k')dk'\,. \tag{2}$$

Usually the energy transfer is thought to occur by creation of small scales through stretching and folding of vortical elements, which is modeled by simplified processes such as the successive breakdown of "eddies" (see, e.g., Ref. 2). One then argues that through scales of motion of size k^{-1}, there is a net flux of kinetic energy to smaller scales, which is equal to $\pi(k)$. In the inertial range[1] one expects this flux to be equal to the average rate of dissipation of kinetic energy $\langle\epsilon\rangle$. However, it is known that in physical space the local rate of dissipation is distributed very intermittently. This can be modeled within the framework of breakdown of eddies, but with the assumption that the flux of energy to smaller scales exhibits spatial fluctuations at every scale (see, e.g., Refs. 2 and 3). Thus we need to define a flux of kinetic energy that, as opposed to Eq. (2), should also depend on position. In general terms then, it is clear that from Fourier spectra any information related to position in physical space is completely hidden, which is a disadvantage when dealing with spatially localized intermittency in the flow. On the other hand, without performing operations involving multiple points, the Navier-Stokes equations in physical space give no explicit information about different scales of motion. Such information is often a useful ingredient for modeling and physical insight. This difficulty calls for a representation that decomposes the flow field into contributions of different scales *as well as* different locations. In other words, we want to use basis functions that behave more like localized pulses than extended waves. If one wishes them to be self-similar, one is led to rather special basis functions, called wavelets.

Wavelets[4] have been used in the field of turbulence, among others, for the study of coherent structures[5] and of two-dimensional data from a turbulent jet.[6] Such studies used nonorthogonal wavelets where the transform typically consists of many more coefficients than the number of points of the original data set. Recently, several orthonormal wavelet basis functions have been constructed,[7,8] and the absence of redundancy of information makes this form of wavelets particularly useful in higher dimensions. In one dimension, orthonormal wavelets are of the form

$$\psi^{(m)}(x-2^mhi)=2^{-m/2}\psi\left(\frac{x-2^mhi}{2^mh}\right)\,, \tag{3}$$

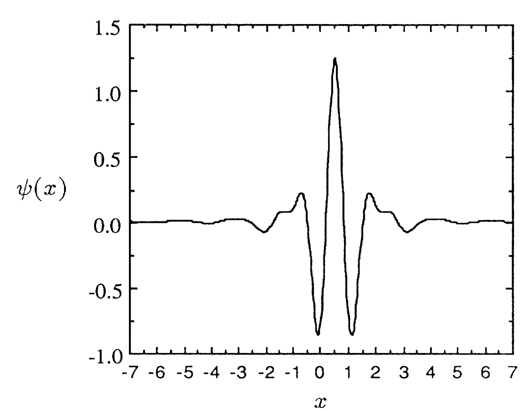

FIG. 1. Example of an orthonormal wavelet $\psi(x)$ (Lemarie-Meyer-Battle wavelet).

where m indicates the octave band of the scale parameter, h is the mesh spacing of the basic lattice (smallest scales), and the index i refers to location in units of $2^m h$. Notice that the basis functions corresponding to the larger scales are spaced more coarsely, according to a dyadic arrangement on a binary tree structure. Some particular functions $\psi(x)$ exhibit the property that $\{\psi^{(m)}(x-2^m hi)\}$ forms an orthonormal base for all (i,m), as, for instance, the Lemarie-Meyer-Battle (LMB) wavelet shown in Fig. 1. It is a real function with exponential decay in x space and k^{-4} decay in Fourier space. For more details on this, on discrete transforms, and on fast algorithms, see Refs. 7 and 8. In an extension of this formalism to three dimensions,[9] the velocity field (at some instant t) is written as

$$u_i(\mathbf{x})=\sum_m \sum_{q=1}^{7} \sum_{(i_1,i_2,i_3)} w^{(m,q)}[\mathbf{i}]\Psi^{(m,q)}(\mathbf{x}-2^m h\mathbf{i}), \tag{4}$$

where m denotes again the scale, $\mathbf{i}=(i_1,i_2,i_3)$ is the three-dimensional position index on a cubic lattice of mesh size $2^m h$, and q gives additional internal degrees of freedom (this is needed because $\Psi^{(m,q)}$ is decomposable into functions in each Cartesian direction). Because of orthonormality the discrete wavelet coefficients can be computed as

$$w_i^{(m,q)}[\mathbf{i}]=\int u_i(\mathbf{x})\Psi^{(m,q)}(\mathbf{x}-2^m h\mathbf{i})d^3x. \tag{5}$$

We now start with the Navier-Stokes equations in physical space written for the fluctuating velocity and pressure and we take its inner product with $\Psi^{(m,q)}(\mathbf{x}-2^m h\mathbf{i})$. Multiplication by $w_i^{(m,q)}[\mathbf{i}]$ and contraction over the three directions i and the index q yields an evolution equation for the local kinetic-energy density $e^{(m)}[\mathbf{i}]=\frac{1}{2}\sum_{i,q}(w_i^{(m,q)}[\mathbf{i}])^2$,

$$\frac{\partial}{\partial t}e^{(m)}[\mathbf{i}]=t^{(m)}[\mathbf{i}]-v^{(m)}[\mathbf{i}], \tag{6}$$

where

$$t^{(m)}[\mathbf{i}]=-\sum_{i=1}^{3}\sum_{q=1}^{7} w_i^{(m,q)}[\mathbf{i}]\int\left(u_j\frac{\partial u_i}{\partial x_j}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right)\Psi^{(m,q)}(\mathbf{x}-2^m h\mathbf{i})d^3x \tag{7}$$

is the net energy transfer to scale $2^m h$ at location $2^m h\mathbf{i}$. $v^{(m)}[\mathbf{i}]$ is the contribution of the viscous terms, including molecular diffusion of energy and dissipation at that scale and location. Equation (6) is the (discrete) analog of Eq. (1), written for the energy of orthonormal pulses rather than waves (see also Kraichnan[10]). In analogy to Eq. (2), the flux of kinetic energy through a spatial region of characteristic size $2^m h$ and location $2^m h\mathbf{i}$ can be computed by adding the transfer density (local transfer divided by the total number of grid points at each scale) over all scales larger than $2^m h$ at that particular location:

$$\pi^{(m)}[\mathbf{i}]=-\sum_{n=m}^{M}2^{3(M-n)}t^{(n)}[\mathbf{j}]. \tag{8}$$

Here M is the scale index of the largest scale considered, and $\mathbf{j}$ (given by the integer part of $2^{m-n}\mathbf{i}$) is the position index of the larger scales (n). Several studies[11-13] have used "wave packets" (essentially wavelets) for obtaining approximations to the Navier-Stokes equations, and have then deduced energy cascade models. Here we actually compute these quantities relevant to the energetics of turbulence, without approximations.

To proceed, we compute the spectral transfer density at wave number $k_m=2\pi/2^m h$ by dividing the total transfer to the band m,

$$\sum_{\mathbf{i}}t^{(m)}[\mathbf{i}]=2^{3(M-m)}\langle t^{(m)}[\mathbf{i}]\rangle,$$

by the bandwidth $\Delta k_m=k_m\ln(2)$ and by the total number of points 2^{3M}. We obtain

$$T_w(k_m)=2^{-3m}k_m^{-1}[\ln(2)]^{-1}\langle t^{(m)}[\mathbf{i}]\rangle, \tag{9}$$

where the average extends over all points $[\mathbf{i}]$. $T_w(k_m)$ is equivalent to the Fourier-transfer spectrum $T(k)$, but is not necessarily identical at every k because of the width of the wavelet in Fourier space. One can now inquire about the spatial variability of $t^{(m)}[\mathbf{i}]$, that can be quantified in terms of its standard deviation [in units of $T(k_m)$] according to

$$\sigma_t(k_m)=2^{-3m}k_m^{-1}[\ln(2)]^{-1}\times(\langle t^{(m)}[\mathbf{i}]^2\rangle-\langle t^{(m)}[\mathbf{i}]\rangle^2)^{1/2}. \tag{10}$$

A plot of $T_w(k_m)$ and $T_w(k_m)\pm\sigma_t(k_m)$ as a function of k_m will be called the *dual spectrum* of transfer, dual because it gives information both about the contribution of various scales, *and* about the spatial variability associated with it. Similar definitions of dual spectra can be introduced for the kinetic energy and the flux of energy.[9]

Next we turn to the analysis of three-dimensional turbulent fields. We consider results of pseudospectral direct numerical simulation of homogeneous sheared turbulence on a 128^3 grid, described in detail in Ref. 14 and references therein. The snapshot considered is at $t=12$ in units of the imposed shear, when the Taylor-scale Reynolds number is about 110. The field is not isotropic, as elongated vortical structures are visible (isotropic turbulence of lower Reynolds number was also considered[9]). We compute the 3D wavelet transform of the three velocity components, using the LMB wavelet basis. To compute the local transfer $t^{(m)}[\mathbf{i}]$ we need to compute the wavelet transform of the nonlinear terms of the Navier-Stokes equation (the pressure is computed from the known fluctuating velocity field by solving the Poisson equation), and then we apply Eq. (7).

Figure 2 shows the dual transfer spectrum $T(k_m)$ and $T(k_m)\pm\sigma_t^{(m)}$ computed from the homogeneous shear flow, in Kolmogorov units. The mean transfer (circles)

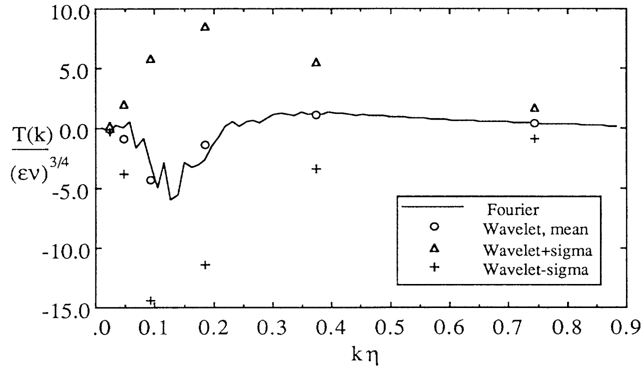

FIG. 2. Dual spectrum of transfer of kinetic energy for homogeneous shear simulation, in Kolmogorov units.

is negative for low wave numbers and positive at high wave numbers, showing that on the average energy is being transferred from large to small scales. The solid line indicates the corresponding radial Fourier-transfer spectrum (obtained from the usual Fourier analysis), in reasonable agreement with the mean wavelet transfer. However, the standard deviation $\sigma_t^{(m)}$ is seen to be very large, implying that locally the transfer of energy is often quite far from its spectral mean value. This is borne out even clearer in the probability-density functions of $t^{(m)}[\mathbf{i}]$ of Fig. 3, which are for three scales $m=1,2,3$. Large deviations away from the mean are visible, both on the positive and negative side. Also, we note the long tails of the distributions, which are of the exponential type.

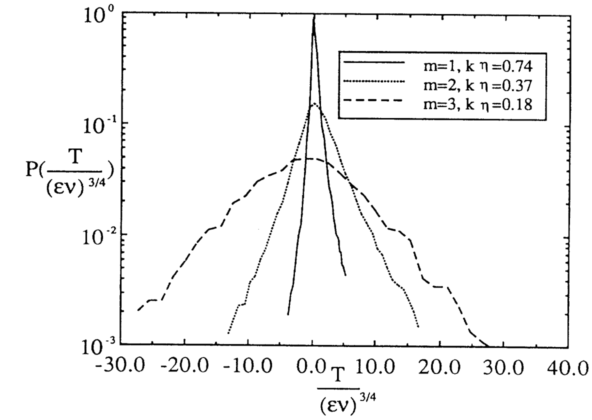

FIG. 3. Probability-density distribution of transfer of kinetic energy at scales $m=1,2,3$.

The quantity $t^{(m)}[\mathbf{i}]$ represents the local transfer through a certain scale without discriminating between the other two scales involved in the nonlinear interactions. In order to define a position-dependent flux of kinetic energy to all scales smaller than some cutoff band m which does not include sweeping by the larger scales, one needs to decompose the nonlinear terms in more detail. By decomposing the velocity into large- and small-scale components, and considering the small-scale part of the pressure such that the large-scale part implies a divergence-free large-scale velocity field, one can show[9] that

$$t^{(m,n)}[\mathbf{i}]=\sum_{i=1}^{3}\sum_{q=1}^{7} w_i^{(m,q)}[\mathbf{i}]\int\left\{\frac{\partial}{\partial x_j}[u_iu_j-u_i^{>n}u_j^{>n}]+\rho^{-1}\frac{\partial}{\partial x_i}p^{<n}\right\}\Psi^{(m,q)}(\mathbf{x}-2^m h\mathbf{i})d^3x\,, \tag{11}$$

represents the transfer of energy between scales m and all scales smaller than n. This quantity is analogous to the Fourier transfer $T(k|k_n)$,[15] defined as the total contribution to $T(k)$ from triads of wave numbers $(k,q,k-q)$ having $k<k_n$ and at least one of the other two legs larger than k_n. In Eq. (11), the superscripts $>n$ $(<n)$ refer to low-pass (high-pass) filtered fields obtained from Eq. (4) by performing the sum over all scales $m\geq n$ $(m<n)$.

A quantity of great practical importance[16] is the effective sink of kinetic energy due to scales of motion smaller than some cutoff. In the Fourier representation, this sink of energy is given[15] as a flux

$$\pi_{\rm sg}(k)=-\int_{k'=0}^{k}T(k'|k)dk'. \tag{12}$$

The analogous definition in the wavelet representation is the local flux of energy to smaller scales,

$$\pi_{\rm sg}^{(m)}[\mathbf{i}]=-\sum_{k=m}^{M}2^{3(M-k)}t^{(k,m)}[2^{k-m}\mathbf{i}]. \tag{13}$$

This quantity is computed from the homogeneous shear flow simulation, its dual spectrum is obtained, and we construct the probability density of this subgrid flux at every scale. The results are shown in Figs. 4 and 5. We make the following observations: The mean subgrid flux is always positive, indicating that on the average, energy

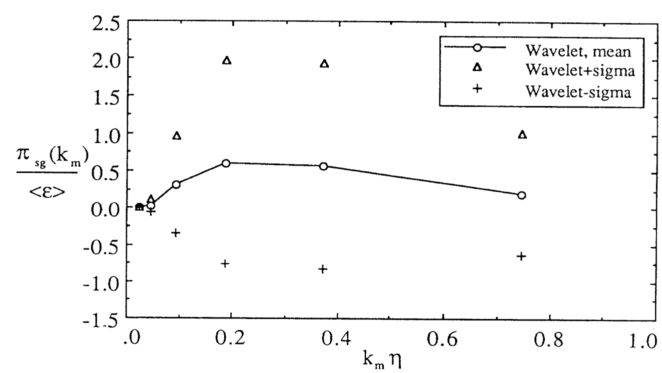

FIG. 4. Dual spectrum of the flux of kinetic energy from interactions with all smaller scales of motion (flux to or from the subgrid scales).

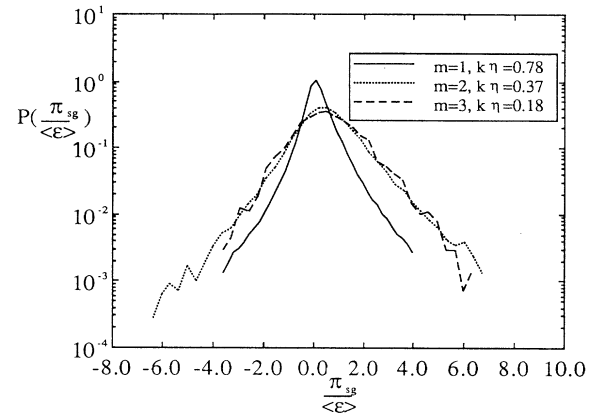

FIG. 5. Probability-density function of spatial fluctuations of local subgrid flux of kinetic energy at scales $m=1,2,3$.

flows from large to small scales. There are strong spatial fluctuations and the statistics of π_{sg} are far from Gaussian, exhibiting very clearly, again, long and exponential tails. These long tails mean that the flux is very intermittent in space, at every scale of motion. The fluctuations of the subgrid flux are such that it can be negative very often (local backscatter). In such locations, energy actually flows from the small to the large scales of motion, i.e., there are local inverse cascades. The tails of the distributions are nearly symmetric to both sides; thus, the average being positive comes from a delicate balance between large positive and not-so-large negative excursions of localized events. The analysis of isotropic decaying turbulence yields similar results[9] and the phenomenon of local backscatter has also been observed recently during analysis (using low-pass filters) of channel-flow simulations.[17] For a stochastic modeling strategy and more references, see Leith.[18]

Although the conclusions have been obtained here for a low-Reynolds-number flow where no fully developed inertial range exists, it seems unlikely that the local backscatter would disappear at higher Reynolds numbers. Many of the phenomenological cascade models of intermittency work under the assumption of local (in x and k space) energy transfer from large to small scales, where, from dimensional arguments, the locally averaged rate of dissipation (always positive) is the quantity representative of the local inertial-range flux of energy. Such a picture must be revised in order to allow for negative fluxes to occur.[9]

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