$B=3$ Nuclei as Quantized Multi-Skyrmions

Larry Carson
Department of Physics and Astronomy, University of Alabama, Box 870324, Tuscaloosa, Alabama 35487
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The minimal-energy solution of the SU(2) Skyrme model with baryon number three is a soliton with tetrahedral symmetry $\bar{4}3m$. Given one assumption, I show how this symmetry ensures that the $J^\pi=\frac{1}{2}^+$ isodoublet nucleus (^{3}He,^{3}H) emerges as the unique ground state of this soliton solution when its isospin and rotational zero modes are quantized.

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In the effective theory of two-flavor strong interactions given by the SU(2) Skyrme model,[1] baryons are described as topological solitons—Skyrmions—in the pion field, where the topologically conserved winding number B of the soliton is identified with its baryon number.[2] The nucleon and Δ have been successfully modeled as the quantum states of the spinning $B=1$ soliton solution, while the deuteron emerges as the lowest-energy state of the quantized $B=2$ toroidal Skyrmion.[3,4] In this Letter, I show that it is also possible to account for the isodoublet nucleus (^{3}He,^{3}H) as the unique ground state of the quantized $B=3$ multi-Skyrmion.

The fundamental object of the SU(2) Skyrme model is the SU(2)-valued chiral field $U(x)$, which is related to the pion fields $\pi^a(x)$, $a=1,2,3$, via

$$U(x)=\exp[2i\pi^a(x)\sigma_a/f_\pi]\,.$$

Here σ_a are the Pauli matrices and f_π is the pion decay constant. Introducing the left-invariant current $L_\mu \equiv U^\dagger \partial_\mu U$, the specific model we shall study is given by the Lagrangian

$$\mathcal{L}=-\frac{f_\pi^2}{16}\mathrm{Tr}L_\mu L^\mu+\frac{f_\pi^2 m_\pi^2}{8}\mathrm{Tr}(U-1)+\frac{1}{32e^2}\mathrm{Tr}[L_\mu,L_\nu][L^\mu,L^\nu]\,, \qquad (1)$$

where m_π is the pion mass and the values $f_\pi=108$ MeV, $e=4.84$ are taken from a fit of the nucleon and Δ masses in this model.[5]

To describe nuclei of baryon number B as chiral solitons, a fundamental condition is that the minimal-energy static solution $U_B(\mathbf{x})$ with corresponding winding B be localized in space and classically stable against fission into solitons of lower baryon number. The $B=2$ toroidal Skyrmion is known to satisfy these properties.[3,6] Recent numerical computations[7] of model (1) formulated on a 60^3 lattice with lattice spacing $a=0.1$ fm have extended the known static solutions satisfying the classical stability criterion from $B=2$ up to $B=6$. If $\mathcal{M}_B$ denotes the classical mass of the soliton solution U_B, then the computations of Ref. 7 give $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3\cong 850$, 1640, 2400 MeV, respectively. In particular, one verifies that $\mathcal{M}_3<\mathcal{M}_2+\mathcal{M}_1<\mathcal{M}_1+\mathcal{M}_1+\mathcal{M}_1$.

One portrayal of the $B=3$ solution $U_3(\mathbf{x})$ is shown in Fig. 1, which displays a surface of constant baryon-number density at the representative value $B^0(\mathbf{x})=0.4$ fm^{-3}, where

$$B^\mu(x)=(\epsilon^{\mu\nu\rho\tau}/24\pi^2)\mathrm{Tr}L_\nu L_\rho L_\tau \qquad (2)$$

is the baryon-number current. This figure is striking in at least two ways: First, the soliton bears no resemblance to a composite of three objects, and second, the baryon-number density possesses a *tetrahedral* symmetry. Upon closer examination, one finds that the pion fields forming U_3 transform according to the irreducible vector representation of the tetrahedral group $\bar{4}3m$. This 24-element discrete group is generated by two elements g and $-h$, which in the vector representation have the canonical form

$$g=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad -h=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}.$$

It is useful to recast these matrices in SU(2) form by in-

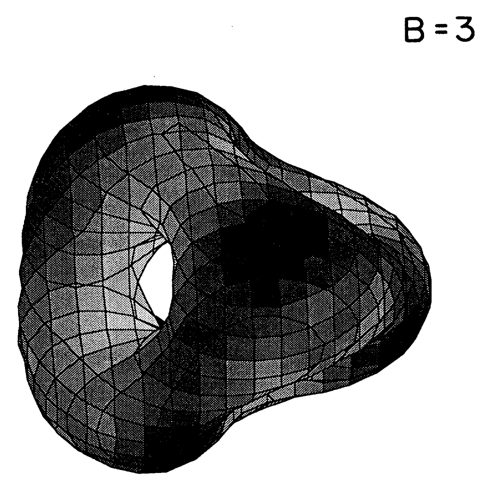

FIG. 1. Surface of constant baryon-number density $B^0(\mathbf{x})=0.40$ fm^{-3} for the $B=3$ multi-Skyrmion solution (from Ref. 7). $B^0(\mathbf{x})$ is tetrahedrally symmetric; in particular, it vanishes along the four rays starting from the center of the soliton and piercing the center of each face of the tetrahedron.

verting the mapping

$$A \to R_{ij}(A) \equiv \tfrac{1}{2}\mathrm{Tr}\sigma_i A \sigma_j A^\dagger, \quad A \in \mathrm{SU}(2).$$

Then $g_{ij} = R_{ij}(G)$ and $h_{ij} = R_{ij}(H)$, where

$$G = \exp\left[i\frac{\pi}{3\sqrt{3}}(\sigma_1+\sigma_2+\sigma_3)\right], \quad H = \exp\left[i\frac{\pi}{4}\sigma_3\right],$$

corresponding to 120° and 90° rotations about the $\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}$ and $\hat{\mathbf{z}}$ axes, respectively. (The signs of G and H are arbitrary, and the choice made here is merely a convenient one.) The tetrahedral symmetry of $U_3(\mathbf{x})$ may then be expressed in the compact form

$$U_3(\mathbf{x}) = G^\dagger U_3(g\cdot\mathbf{x})G = H^\dagger U_3(-h\cdot\mathbf{x})^\dagger H. \qquad (3)$$

Semiclassical quantization of the multi-Skyrmion proceeds by promoting the isospin and rotational zero modes of the soliton, represented by SU(2) matrices A and A', to dynamical degrees of freedom. Making the replacement

$$U_3(\mathbf{x}) \to \hat{U}_3(\mathbf{x},t) = A(t)U_3(R(A'(t))\cdot\mathbf{x})A(t)^\dagger, \qquad (4)$$

and inserting this into (1), we obtain a reduced Lagrangian $L_{\rm red}$ quadratic in the time derivatives $a_k = -i \times \mathrm{Tr}\sigma_k A^\dagger \dot{A}$ and $b_k = i\,\mathrm{Tr}\sigma_k \dot{A}' A'^\dagger$:

$$L_{\rm red} = \int d^3x\, \mathcal{L}_{\rm red} = \tfrac{1}{2} a_i U_{ij} a_j - a_i W_{ij} b_j + \tfrac{1}{2} b_i V_{ij} b_j - \mathcal{M}_3.$$

Here the inertia tensors U_{ij}, V_{ij}, and W_{ij} are certain functionals of the background field $U_3(\mathbf{x})$. (For the explicit expressions of these tensors and other details of the quantization method used here, see Ref. 4.) This reduced Lagrangian may now be quantized by the usual techniques; a_j and b_j are exchanged for the body-fixed isospin and spin angular momentum operators K_j and L_j, canonically conjugate to A and A'. These operators are related to the usual coordinate-fixed isospin and spin angular momentum operators I_j and J_j via

$$I_i = -R_{ij}(A)K_j, \quad J_i = -R_{ij}(A')^T L_j,$$

implying $\mathbf{I}^2 = \mathbf{K}^2$ and $\mathbf{J}^2 = \mathbf{L}^2$. The twelve operators $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{L}$ generate in fact the Lie algebra of $\mathrm{O}(4)_{I,K} \otimes \mathrm{O}(4)_{L,J}$. Their action on the coordinates A and A' is given by

$$[I_i, A] = -\tfrac{1}{2}\sigma_i A, \quad [K_i, A] = \tfrac{1}{2} A\sigma_i,$$
$$[L_i, A'] = -\tfrac{1}{2}\sigma_i A', \quad [J_i, A'] = \tfrac{1}{2} A'\sigma_i, \qquad (5)$$

while all other commutators between momenta and coordinates vanish. The Hilbert space is spanned by the states $|I, I_3, K_3; J, J_3, L_3\rangle$, where $-I \le I_3, K_3 \le I$ and $-J \le J_3, L_3 \le J$. The subspace of fixed I, I_3, J, and J_3, labeled by the states $|K_3, L_3\rangle$, is $(2I+1)(2J+1)$ dimensional.

The space of physical states is also restricted by the requirements of the Pauli exclusion principle, which is implemented here in the form of Finkelstein-Rubinstein-Williams (FRW) constraints.[8] The latter amount to the requirement that states of isolated $B=1$ Skyrmions be quantized as fermions, which in particular pick up a phase of -1 when adiabatically rotated by 2π. However, it turns out that FRW constraints may be associated with any one-parameter set of static finite-energy fields, $U(\mathbf{x},\theta)$, $0 \le \theta \le 2\pi$, which is closed: $U(\mathbf{x},2\pi) = U(\mathbf{x},0)$. These define maps from S^4 into SU(2) describing loops in configuration space that are either contractible or noncontractible according to the homotopy $\pi_4(\mathrm{SU}(2)) = Z_2$. Transformations of states associated with noncontractible loops, like the 2π rotation of a $B=1$ Skyrmion, are assigned a phase -1 while those associated with loops that are contractible to a point (no transformation) must be assigned the phase $+1$.

For $B=3$, isospin and spatial rotations by 2π are individually noncontractible.[9] Hence for any unit vector $\hat{\mathbf{n}}$,

$$e^{2\pi i \hat{\mathbf{n}}\cdot\mathbf{K}}|\mathrm{phys}\rangle = e^{2\pi i \hat{\mathbf{n}}\cdot\mathbf{L}}|\mathrm{phys}\rangle = -|\mathrm{phys}\rangle,$$

implying that K and L, and thus I and J, are half integral. From the symmetries (3) one can construct two additional closed loops and their associated FRW constraints. The requirement that the loops be closed will restrict us to the proper subgroup 23 of the full tetrahedral group $\bar{4}3m$, generated by $f = (-h)^2$ and g. Then defining $\mathbf{M} = \mathbf{K} + \mathbf{L}$,

$$\hat{U}_3(\mathbf{x},\theta_f) = e^{-i\theta_f M_3}\hat{U}_3(\mathbf{x})e^{i\theta_f M_3}, \quad 0 \le \theta_f \le \pi, \qquad (6)$$

$$\hat{U}_3(\mathbf{x},\theta_g) = \exp[-i\theta_g(M_1+M_2+M_3)/\sqrt{3}]\hat{U}_3(\mathbf{x}) \times \exp[i\theta_g(M_1+M_2+M_3)/\sqrt{3}], \qquad (7)$$
$$0 \le \theta_g \le 2\pi/3,$$

are the loops of interest. From Eqs. (3)–(5), one can show these loops are closed:

$$\hat{U}_3(\mathbf{x}) = \hat{U}_3(\mathbf{x},\theta_f=0) = \hat{U}_3(\mathbf{x},\theta_f=\pi) = \hat{U}_3(\mathbf{x},\theta_g=0) = \hat{U}_3(\mathbf{x},\theta_g=2\pi/3).$$

The contractibility of loop (7) is easily deduced by observing that this loop traversed 3 times is equivalent to an isorotation by 2π *plus* a spatial rotation by 2π. By the multiplication rules of Z_2, the product of two noncontractible loops is contractible, and so we infer

$$\exp\left[-i\frac{2\pi}{3\sqrt{3}}(M_1+M_2+M_3)\right]|\mathrm{phys}\rangle = +|\mathrm{phys}\rangle. \qquad (8)$$

This constraint is most easily solved in the basis provided by the eigenstates $|M, M_3\rangle$ of $\mathbf{M}$ angular momentum. For $M = 0$, 1, and 2, the unique states satisfying (8) are, respectively,

$$|0,0\rangle\,, \tag{9}$$

$$|1,1\rangle\frac{z^*}{\sqrt{3}}-|1,0\rangle\frac{1}{\sqrt{3}}-|1,-1\rangle\frac{z}{\sqrt{3}}\,, \tag{10}$$

$$|2,2\rangle\frac{(z^*)^2}{\sqrt{6}}-|2,1\rangle\frac{z^*}{\sqrt{3}}+|2,-1\rangle\frac{z}{\sqrt{3}}+|2,-2\rangle\frac{z^2}{\sqrt{6}}\,, \tag{11}$$

where $z=e^{i\pi/4}$. In deriving (9)-(11), I have used Condon-Shortley phase conventions for the Wigner D functions.[10]

The (non)contractibility of loop (6) cannot be deduced by the method I used above for (7). Thus we do not know *a priori* whether the end-point operator $e^{-i\pi M_3}$ acting on physical states gives ± 1. In the absence of mathematical proof, I shall fix the phase consistent with the known existence of the $I=J(=K=L)=\frac{1}{2}$ isodoublet nucleus (^{3}He,^{3}H). By the addition rules of angular momenta, this state must be a linear combination of states with $M=0$ and 1, and hence of states (9) and (10). But only the first of these is an eigenstate of $e^{-i\pi M_3}$, with eigenvalue $+1$. Therefore,

$$e^{-i\pi M_3}|\text{phys}\rangle=+|\text{phys}\rangle\,. \tag{12}$$

This excludes both (10) and (11) from the space of physical states, leaving (9) as the unique state with $I=J=\frac{1}{2}$. This also implies that there are no states with the quantum numbers $I=\frac{1}{2}$, $J=\frac{3}{2}$ or $I=\frac{3}{2}$, $J=\frac{1}{2}$.

The two FRW constraints (8) and (12) may be summarized as follows: Physical states must transform as the *trivial* representation of the proper tetrahedral group 23 as embedded in $SU(2)_M$. For $I,J\le\frac{3}{2}$, the only allowed states are

$$I=J=\tfrac{1}{2}:|0,0\rangle\,, \tag{13}$$

$$I=J=\tfrac{3}{2}:|0,0\rangle\,, \tag{14}$$

$$I=J=\tfrac{3}{2}:|3,2\rangle(z^*)^2/\sqrt{2}+|3,-2\rangle z^2/\sqrt{2}\,, \tag{15}$$

again working in the $|M,M_3\rangle$ basis. The parity of these states are obtained as the eigenvalues of an operator $\mathcal{P}$ which takes

$$\hat{U}_3(\mathbf{x})\to\mathcal{P}\hat{U}_3(\mathbf{x})\mathcal{P}^\dagger=\hat{U}_3(-\mathbf{x})^\dagger\,,$$

as deduced from the negative intrinsic parity of the pion field. Using (3)-(5), one finds that $\mathcal{P}=e^{i\pi M_3/2}$, so that states (13)-(15) have $+$, $+$, and $-$ parity, respectively.

The masses of allowed states are computed as expectation values of the Hamiltonian operator $\hat{H}$ derived from L_{red}. The symmetries (3) imply that the inertia tensors are all proportional to the unit matrix, e.g., $U_{ij}=u\delta_{ij}$, $V_{ij}=v\delta_{ij}$, and $W_{ij}=w\delta_{ij}$, where u, v, and w are respectively found to be 136, 435, and -91 in units of $1/e^3f_\pi$. Expressing $\hat{H}$ in terms of momentum operators, one finds

$$\hat{H}=\mathcal{M}_3+\frac{1}{2}\frac{1}{uv-w^2}[(v-w)\mathbf{I}^2+(u-w)\mathbf{J}^2+w\mathbf{M}^2]\,.$$

Evaluating for states (13)-(15) above, we obtain

$$\langle\tfrac{1}{2},\tfrac{1}{2}^+|\hat{H}|\tfrac{1}{2},\tfrac{1}{2}^+\rangle=\mathcal{M}_3+\frac{3}{8}\frac{u+v-2w}{uv-w^2}=2464\text{ MeV}\,,$$

$$\langle\tfrac{3}{2},\tfrac{3}{2}^+|\hat{H}|\tfrac{3}{2},\tfrac{3}{2}^+\rangle=\mathcal{M}_3+\frac{15}{8}\frac{u+v-2w}{uv-w^2}=2735\text{ MeV}\,,$$

$$\langle\tfrac{3}{2},\tfrac{3}{2}^-|\hat{H}|\tfrac{3}{2},\tfrac{3}{2}^-\rangle=\mathcal{M}_3+\frac{3}{8}\frac{5u+5v+6w}{uv-w^2}=2604\text{ MeV}\,.$$

From its mass splitting (approximately 270 MeV) with the $J^\pi=\frac{1}{2}^+$ ground state and its quantum numbers, the $\frac{3}{2}^+$ state may be interpreted as an $NN\Delta$ nucleus in which one nucleon is excited to a Δ isobar. Such excitations have been seen, for instance, in inelastic electron scattering off ^{3}He nuclei (cf. Ref. 11). On the other hand, while the quantum numbers of the $\frac{3}{2}^-$ state are also accountable by a trinucleon system in which one nucleon is excited [e.g., $NNN(1520)$], its excitation energy of 140 MeV is simply too small for this interpretation. Similar anomalously low-mass states appear in the spectrum of dibaryon resonances of the quantized bi-Skyrmion,[4] but their excitation energies increase once nonzero modes of the soliton are quantized.[12] Further work, therefore, will be required to accurately determine the mass of this tribaryon resonance.

In summary, by making one assumption, namely, the contractibility of loop (6), I have shown that the ground state of the quantized $B=3$ multi-Skyrmion is the unique state with the quantum numbers $I=\frac{1}{2}$ and $J^\pi=\frac{1}{2}^+$. To test the identification of this state with the physical ^{3}He and ^{3}H nuclei, a computation of its static electromagnetic properties (magnetic moments, charge radii, etc.) is now in progress. The results of that investigation will be reported in a forthcoming publication.

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[1]T. H. R. Skyrme, Proc. Roy. Soc. London A **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).

[2]For reviews, see A. P. Balachandran, in *High Energy Physics 1985,* edited by M. J. Bowick and F. Gürsey (World Scientific, Singapore, 1985); I. Zahed and G. E. Brown, Phys. Rep. **142**, 1 (1986); G. Holzwarth and B. Schwesinger, Rep. Prog. Phys. **49**, 825 (1986); J. Wambach, in Proceedings of the Lectures Given at the Spring School of Medium- and High-Energy Nuclear Physics, Taiwan, May 1990 (to be published).

[3]J. J. M. Verbaarschot, T. S. Walhout, J. Wambach, and H. W. Wyld, Nucl. Phys. **A468**, 520 (1987).

[4]E. Braaten and L. Carson, Phys. Rev. D **38**, 3525 (1988).

[5]G. S. Adkins and C. R. Nappi, Nucl. Phys. **B233**, 109 (1984).

[6]V. B. Kopeliovich and B. E. Shtern, Zh. Eksp. Teor. Fiz. **45**, 165 (1987) [JETP Lett. **45**, 203 (1987)]; A. J. Schramm, Y. Dothan, and L. C. Biedenharn, Phys. Lett. B **205**, 151 (1988).
[7]E. Braaten, S. Townsend, and L. Carson, Phys. Lett. B **235**, 147 (1990).
[8]D. Finkelstein and J. Rubinstein, J. Math. Phys. **9**, 1762 (1968); J. G. Williams, J. Math. Phys. **11**, 2611 (1970).
[9]E. Witten, Nucl. Phys. **B233**, 433 (1983).
[10]See, for example, K. Gottfried, *Quantum Mechanics* (Benjamin, Reading, MA, 1974), Secs. 32 and 34.
[11]D. Day *et al.*, Phys. Rev. Lett. **43**, 1143 (1979).
[12]J. Wambach (private communication).