Does Quantum Mechanics Violate the Bell Inequalities?

Emilio Santos
Departamento de Física Moderna, Universidad de Cantabria, Santander, Spain
(Received 17 August 1990; revised manuscript received 4 September 1990)

It is argued that any quantum probability should correspond to a ratio between the number of counts (positive results) in some measurement and the number of copies of the physical system initially prepared. Then, the proofs of Bell's theorem are criticized on the grounds that the probabilities used to show a violation of the Bell inequalities do not fulfill that condition. A hidden-variables model is proposed which reproduces the results of the optical experimental tests of the inequalities, even with perfect polarizers and detectors.

PACS numbers: 03.65.Bz, 42.50.Wm

Bell's theorem[1] states that local hidden-variables theories of quantum mechanics are not possible. The proof consists of two parts. First, some inequalities are derived which should be fulfilled by every local hidden-variables theory. Second, it is shown that there are quantum-mechanical predictions violating the inequalities. The purpose of this Letter is to point out that the second step is incorrect in the existing proofs and, consequently, the theorem has not yet been proved.

Although I have no criticism of the derivation of the Bell inequalities, it is convenient to recall the steps in the derivation, following Clauser and Horne.[2] We consider an EPR- (Einstein-Podolsky-Rosen-) type experiment in which we prepare a pair of correlated particles, and we measure a dichotomic (yes-no) observable on each particle, the measurement events having spacelike separation. Let $p(a_1,b_2)$ be the joint probability of getting the answer yes in both measurements. Then, on an identically prepared two-particle system, we measure another pair of dichotomic observables, obtaining the probability $p(c_1,d_2)$. In a similar way we obtain $p(a_1,d_2)$ and $p(c_1,b_2)$. In the first or in the third experiment, we measure the probability $p(a_1)$ of getting the answer yes for the observable a_1 of the first particle, independently of the result of the measurement on the second particle [the value obtained for $p(a_1)$ should be the same whether it is measured in the first or in the third experiment]. Similarly, we measure $p(b_2)$ either in the first or in the fourth experiment. Under these conditions, any local hidden-variables model predicts the following inequality:[2]

$$p(a_1)+p(b_2)\geq p(a_1,b_2)+p(a_1,d_2)+p(c_1,b_2)-p(c_1,d_2)\,. \tag{1}$$

In the proofs of Bell's theorem, the contradiction with the inequality (1) (or another similar one) is usually shown by considering a spin-correlated pair of particles placed at two points $\mathbf{x}_1$ and $\mathbf{x}_2$ (or within small regions around these points) at the same time t (time measured in some inertial frame, e.g., the laboratory). For instance, if the particles are photons in a singlet state, we may choose as observables the polarizations in four directions, all perpendicular to the vector $\mathbf{x}_2-\mathbf{x}_1$. Let $|\psi\rangle$ be the Hilbert-space vector corresponding to the two-photon state as defined above, and A_1,C_1,B_2,D_2 the four polarization operators. Then, the probabilities involved in (1) are

$$p(a_1)=\langle\psi|A_1|\psi\rangle=\tfrac{1}{2}\,,\qquad p(a_1,b_2)=\langle\psi|A_1B_2|\psi\rangle=\tfrac{1}{4}[1+\cos2(a-b)]\,, \tag{2}$$

and similar results for the other cases, $(a-b)$ being the angle between the corresponding directions of polarization. These probabilities violate the inequality (1) for some choices of the polarizations, which completes the usual proof of Bell's theorem.

My criticism to the proof is that the two-particle state defined above, represented by the Hilbert-space vector $|\psi\rangle$, has not been shown to be a physically realizable state. Of course, it may be postulated that *every* vector in the Hilbert space of the system corresponds to a physically realizable state. But it is by no means obvious that such a strong assumption should be a part of the quantum formalism. In fact, it is well known that some restrictions have already been introduced under the name of superselection rules, and further constraints may be found in the future. Consequently, I claim that any proof of violation of a Bell inequality by quantum predictions should necessarily involve finding an (ideal) experiment with the following conditions. We should prepare a physical system in some region of space and allow it to evolve. The system is divided into two (or more) subsystems, each going to a different region of space. Finally, we should get the correlation between appropriate observables by means of two spatially separated measurements. In the full experiment many copies of the initial system should be identically prepared and the fraction of times that the measurement gives a positive answer is to be used as (an approximation of) the probability suitable for checking Eq. (1).

The point is that only the ratio between the final number of counts measured and the initial number of systems prepared corresponds to a genuine probability. Other numbers which may appear at intermediate steps of the calculation should not be interpreted as probabilities, no matter how appealing the interpretation may appear to our intuition. If we insist on such interpretations, we may even obtain negative probabilities, as correctly pointed out by Feynman.[3]

In order to clarify the point, I shall study the tests of Bell's inequalities performed by measuring the polarization correlation of optical photon pairs. These form the main class of experiments where it is generally believed that a Bell inequality has been violated. For the sake of clarity, I shall consider tests using 0-1-0 atomic cascades and one-channel polarizers, but the analysis for other cases leads to similar conclusions.

If we want to prepare a two-photon system in a singlet spin state, we must do that by taking an atom localized around some point $\mathbf{x}$ and having total angular momentum $J=0$. If the atom emits two photons in a cascade, passing through a state of $J=1$ towards a state $J=0$, then the pair of emitted photons has zero total angular momentum. This was the procedure used in the celebrated experimental tests of Bell's inequalities by Aspect and co-workers.[4] But the state so prepared is quite different from the one considered in relation to Eq. (2). The latter, having zero angular momentum, is a state of spherical symmetry, which means that the photons are not localized but spread in the form of a spherical wave. Consequently, the probability for one photon being detected by an ideal detector placed in the direction of the unit vector $\mathbf{u}_1$, or both photons simultaneously detected by two perfect detectors placed in the directions $\mathbf{u}_1$ and $\mathbf{u}_2$, is, respectively,

$$P(\mathbf{u}_1)=\langle\psi|U_1|\psi\rangle=\Omega/4\pi\,, \tag{3}$$

$$P(\mathbf{u}_1,\mathbf{u}_2)=\langle\psi|U_1U_2|\psi\rangle=(\Omega/4\pi)^2\alpha(\theta,\varphi) \tag{4}$$

($\cos\theta\equiv\mathbf{u}_1\cdot\mathbf{u}_2$), where Ω is the solid angle covered by the apertures of the lens systems (assumed the same for both photons, for the sake of simplicity), U_1 is the quantum projection operator corresponding to the observable localization of the photon in the given solid angle, and similarly for U_2. The half-angle φ of the cone covered by the apertures is related to the solid angle Ω by

$$\Omega=2\pi(1-\cos\varphi)\,. \tag{5}$$

Finally, $\alpha(\theta,\varphi)$ is an appropriate angular correlation function which I exhibit here in the most relevant cases

$$\begin{aligned}\alpha(\theta,0)&=\tfrac{3}{4}(1+\cos^2\theta)\,,\\ \alpha(\pi,\varphi)&=1+\tfrac{1}{8}\cos^2\varphi(1+\cos\varphi)^2\,.\end{aligned} \tag{6}$$

The first expression is the well-known angular correlation for 0-1-0 cascades and point detectors. The second one can be obtained from the first by a straightforward integration when the solid angles covered by the apertures are finite.

Similarly, if we insert appropriate ideal polarizers in front of the detectors, the single and joint probabilities for detection are

$$p(\mathbf{u}_1a_1)=\langle\psi|U_1A_1|\psi\rangle=\Omega/8\pi\,, \tag{7}$$

$$\begin{aligned}p(\mathbf{u}_1a_1,\mathbf{u}_2b_2)&=\langle\psi|U_1A_1U_2B_2|\psi\rangle\\ &=(\Omega/8\pi)^2\alpha(\theta,\varphi)[1+F(\theta,\varphi)\cos2(a-b)]\,,\end{aligned} \tag{8}$$

where the so-called depolarization factor[5] $F(\theta,\varphi)$ is related to the change in polarization correlation when two photons have wave vectors making an angle different from π. The expression for $F(\theta,\varphi)$ is cumbersome,[5] but for $\mathbf{u}_2=-\mathbf{u}_1$ (i.e., $\theta=\pi$) and small φ, it is given by

$$F(\pi,\varphi)\approx 1-\tfrac{2}{3}(1-\cos\varphi)^2\,, \tag{9}$$

which is very good approximation even for $\varphi=\pi/6$.

The important point that I want to stress is that (3), (4), (7), and (8) are correct quantum probabilities, obtained by means of the standard quantum rules for the calculation of expectation values (for observables with range $\{0,1\}$, as used here, probabilities are the same as expectations). In sharp contrast, expressions (2) are not expectations in a physical quantum-mechanical state. Therefore (7) and (8), but not (2), should be used to check whether the quantum predictions violate the Bell inequality (1), and the result is that they do not. In fact, putting (7) and (8) into (1), it can be checked that the right-hand side is smaller than the left-hand side whenever the following inequality holds:

$$(1+\sqrt{2}F)\alpha\Omega<8\pi\,. \tag{10}$$

Using Eqs. (5) and (6) and the fact that $F\le 1$ it can be checked that this inequality is always fulfilled in the atomic cascade experiments considered here.

The use of (2) has been sometimes justified by saying that the probabilities involved in the Bell inequality can be defined for "the ensemble of photon pairs such that both members of the pair enter the corresponding apertures." Such an ensemble, however, does not correspond to a quantum state, but involves the implicit assumption that it makes sense to speak about whether a photon has passed through the apertures. This amounts to making statements about the positions of the photons at times intermediate between the preparation and the measurement, without actually measuring these positions. (Observe the similarity with asking by which slit a photon passes in the two-slit experiment.) This is clearly foreign to the quantum formalism. It must also be taken into account that the two-photon state with zero angular momentum is a pure state according to quantum mechanics and, therefore, should not be considered as composed of several distinguishable subensembles.

The Bell inequalities are not sufficient conditions for the existence of local hidden-variables models. Therefore the compatibility of quantum predictions with Bell inequalities does not prove that such models actually exist. A proof would consist of constructing a specific model, but such a model would be rather complex if we want to reproduce the quantum predictions (7) and (8) with the full dependence in θ in φ. Here we exhibit only a simplified model in which we ignore the angular correlation, putting $\alpha=1$, and restrict ourselves to the particular (but most important) case $\theta=\pi$. We consider two hidden variables λ_1 and λ_2 with values in the interval $[0,\pi]$ and define an ensemble of "photon pairs" by the density

$$\rho(\lambda_1,\lambda_2)=[1+\cos(2\lambda_1-2\lambda_2)]/\pi^2 . \tag{11}$$

We assume that the probability that a photon with hidden variable λ is detected after crossing a polarizer set at a ($a \in [0,\pi]$) is given by

$$P(\lambda,a)=\begin{cases}\beta \text{ if } |\lambda-a| \leq \gamma(\text{mod}\pi), \\ 0 \text{ otherwise.}\end{cases} \tag{12}$$

Putting (11) and (12) in the standard Bell's formulas for single and joint probabilities

$$p(a_1)=\int \rho(\lambda_1,\lambda_2)P(\lambda_1,a_1)d\lambda_1 d\lambda_2 , \tag{13}$$

$$p(a_1,b_2)=\int \rho(\lambda_1,\lambda_2)P(\lambda_1,a_1)P(\lambda_2,b_2)d\lambda_1 d\lambda_2 , \tag{14}$$

we obtain (7) and (8) provided we choose

$$(2\gamma)^{-1}\sin 2\gamma=\sqrt{F}, \quad \beta\gamma=\Omega/16 . \tag{15}$$

For solid angles not too large this gives, taking (5) and (9) into account,

$$\gamma \cong \sqrt{2}\Omega/4\pi, \quad \beta \cong \sqrt{2}\pi/8 , \tag{16}$$

where β is smaller than 1 as it should be [see (12) and remember that P is a probability]. In contrast, the model cannot reproduce the result (2) without violating the condition $P(\lambda,a) \leq 1$. Finally, putting $P=\Omega/4\pi$, instead of (12), we can also reproduce (3) and (4).

I emphasize that the model does *not* rest upon the low efficiency of optical photon detectors, as previous models do.[2,6] Polarizers as well as detectors have been assumed ideal. The model exploits the fact that the depolarization factor $F(\pi,\varphi)$ is smaller than 1 and decreases with increasing φ [see Eq. (9)]. Depolarization is usually considered as a minor practical effect of the nonideality of the experiments. But it is not so. It is a physical phenomenon related to the fact that linear polarization of a photon is only defined in directions perpendicular to the wave vector, which in turn is a consequence of the transversality of the electromagnetic waves.

A final comment is in order about the empirical tests of the Bell inequalities performed by Aspect and coworkers[4] and others.[6] The received wisdom is that the inequalities are violated not only by quantum mechanics, but they have been actually violated by the experiments. The fact is that the experiments have confirmed Eqs. (3)-(8), and therefore quantum mechanics, but the claimed violation of the Bell inequalities arises only when the quotients of (7) to (3) and (8) to (4), measured as ratios of counting rates, are incorrectly interpreted as probabilities. In fact, true probabilities in these experiments should correspond to ratios of counting rates to preparation rates (say, decay rates in the source). In contrast, the ratio between two counting rates in two different experiments (one with polarizers in place, and the other with the polarizers removed) is not a probability. It is just a ratio of probabilities provided we are sure that the decay rate in the source was the same in both experiments. Certainly, the identification is made more palatable by showing the violation by the experiments of some inequalities involving only ratios of probabilities. However, the new inequalities are not mere consequences of locality, but need additional assumptions.[6] Therefore the experiments have only refuted those local hidden-variables models which agree with these additional hypotheses. A similar criticism applies to the recent experiment by Rarity and Tapster.[7] In fact, the correlation coefficient defined by their Eq. (1) is a ratio of detection rates rather than a quotient of some combination of detection rates by a production rate.

I acknowledge the financial support of Direccion General de Investigacion Cientifica y Tecnica Project No. PB-87-0014 (Spain).

[1]J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1965).

[2]J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).

[3]R. P. Feynman, Int. J. Theor. Phys. **21**, 467 (1982).

[4]A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. **47**, 460 (1981); **49**, 91 (1982); A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).

[5]J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[6]*Quantum Mechanics versus Local Realism. The Einstein-Podolsky-Rosen Paradox,* edited by F. Selleri (Plenum, New York, 1988).

[7]J. G. Rarity and P. R. Tapster, Phys. Rev. Lett. **64**, 2495 (1990).