# Wess-Zumino Term for Chiral Bosons

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(Received 7 May 1990)

We derive the Wess-Zumino term for the chiral-boson model formulated by Sonnenschein using an iterative process, previously introduced by the author in the context of the chiral Schwinger model, that transforms the second-class chiral constraints into first-class ones. Because of the peculiar features of this model the Wess-Zumino term found is made of an infinite number of terms, using an infinite number of auxiliary fields.

PACS numbers: 11.10.Ef, 03.70.+k, 11.10.Lm, 11.30.Rd

Self-dual fields in two space-time dimensions, sometimes called chiral bosons, have received much attention in the past few years, originally because of their major role in the formulation of heterotic strings,[1] and more recently because of the observation by Wen that the gapless boundary excitations of a quantum Hall liquid, the so-called "edge states," form a representation of a chiral Kac-Moody algebra.[2] The existence of these excitations is a characteristic property of the fractional-quantum-Hall-effect states, which can be tested experimentally. Subsequently Stone[3] showed that these algebras may be represented by chiral-bosonic fields describing fluctuations in the shape of a two-dimensional electron gas, called "ripplons." Ripplons are the edge degree of freedom of a Chern-Simons action with dynamical fields. An interesting interface between solid-state and particle physics is bringing all those beautiful mathematical constructions of string theory into the reach of experimental investigation, in the domain of the quantum Hall effect. For example, the central charges of Kac-Moody algebras are related to the electric charge of the edge excitations, while the unidirectional motion of the edge states provides a concrete realization of heterotic strings.

However, the quantization of these fundamental objects has been beset with notorious difficulties in the Lagrangian formulation, while the left- or right-moving scalar fields are easily described in the Hamiltonian language. The usual route has been to start with a Lagrangian for a scalar field and project out the undesired component by means of chiral constraints $\partial_+\phi \approx 0$, which actually only serves to complicate matters. In order for the Lagrangian multiplier not to become dynamical, one needs the constraint to be first class, which is hardly the case here. To circumvent these problems two proposals for the construction of chiral bosons appeared. First, Siegel[4] proposed to set one of the components of the energy-momentum tensor to zero, resulting in an action with the second-class chiral constraints squared, which has a gauge symmetry, called "Siegel's symmetry." However, since the (squared) constraint is proportional to the $(--)$ component of the energy-momentum tensor, Siegel's symmetry becomes anomalous at the quantum level on account of the central extension of the algebra of these objects. Therefore the constraint becomes second class again after quantization.[5] To fix this, it has been suggested to use 26 chiral bosons[6] or to include a compensator Liouville action[7] which cancels the anomaly. This scheme, however, has failed to give the correct gravitational anomaly when coupled to gravity. The second approach suggests the use of Dirac brackets to deal with the second-class nature of the constraint.[8] Following this approach, Srivastava[9] showed that the use of a linear constraint does not render the Lagrange multiplier dynamical but his solution has been criticized by Harada[10] as violating unitarity. Meanwhile, Floreanini and Jackiw[11] offered a solution for a single self-dual field in three different ways: by a nonlocal Lagrangian in terms of local fields, by a local Lagrangian in terms of nonlocal fields, and by a local Lagrangian described by local fields but which are of fermionic character. Bernstein and Sonnenschein then showed that Floreanini and Jackiw's solution is equivalent to Siegel's action.

Sonnenschein[12] proposed a non-Abelian generalization for the chiral bosons. To work in this extended scenario before specializing to the Abelian case is indeed the best approach to understand the physics of this system. This model does not suffer from any obstruction to quantization, such as the gravitational anomaly, has the correct equations of motion, and only one Kac-Moody algebra.[13] A geometrical construction of Sonnenschein's chiral boson has been offered by Stone.[14] In this Letter we will follow Stone's route to construct the U(1) version of Sonnenschein's chiral boson. The resulting action is singular and has a second-class primary chiral constraint.[15] In order to reestablish the gauge symmetry we apply an algorithm developed earlier by the present author,[16] in the context of the chiral Schwinger model. The basic idea there was the introduction of appropriated counterterms into the action which turns the constraints first class. In the eventuality that new second-class constraints (called virtuals) appear during this process one has to repeat the operation, as many times as necessary, until the set of constraints is rendered first class. As we are going to show below, for the case of Sonnenschein chiral boson this loop process never ends and as a result the Wess-Zumino term for this model contains an infinite number of terms, described by an infinite number of auxiliary Wess-Zumino fields. The reason for this peculiar behavior lies in the fact that each time a counterterm is introduced to turn the set of constraints first class, a new virtual constraint is generated, which is an exact copy of the original second-class one. Because of this viral-like constraint replication one cannot stop the constraint's conversion process at any finite step. However, with the inclusion of the "final" Wess-Zumino term, the constraint's set is first class and quantization can be achieved by the usual means, as, for example, using the formalism developed by Batalin, Fradkin, and Vilkovisky,[17] as will be mentioned later.

In fact, the simplest route to chiral bosonization is through the quantum-mechanical concept of coherent-state path integrals. These can be achieved starting with an irreducible representation $D(g)$ of some continuous group $G$. A collection of generalized coherent states, labeled by $g \in G$, is

$$|g\rangle = D(g)|0\rangle \,, \tag{1}$$

where $|0\rangle$ is the highest-weight state. Because of the irreducibility of the representation, Schur's lemma shows that these states satisfy an (over)completeness relation. One then uses the overcompleteness property of these states to give a path-integral representation for the vacuum persistence amplitude

$$Z = \mathrm{Tr}(e^{-itH}) \tag{2}$$

by repeatedly inserting the resolution of the identity into (2). Following Stone[14] one finds that the action in the path integral is given by

$$S(g) = -\frac{1}{4\pi}\int d^2x\, \mathrm{tr}(g^{-1}\partial_x g)^2 + \frac{1}{2\pi}\int d^2x\, d\tau\, \mathrm{tr}[g^{-1}\partial_t g\, \partial_x(g^{-1}\partial_\tau g)] \,, \tag{3}$$

where we have extended the group functions $g(x,t)$ to $g(x,t,\tau)$ defined in the interior of a region bounded by the two-dimensional space-time. The classical equation of motion is

$$\partial_x(g^{-1}\partial_x g + g^{-1}\partial_t g) = 0 \,, \tag{4}$$

whose solution reads

$$g(x,t) = g_1(x-t)g_2(t) \tag{5}$$

with $g_2(t)$ an arbitrary group-valued function depending on time only. These solutions look like right-going waves, but in addition to that there is a hidden gauge symmetry which manifests itself in the factor $g_2(t)$. This symmetry has its origin in the invariance of the action under the transformation $g \to gh$, where $h \in H$ leaves $|0\rangle$ fixed up to a phase, which reduces the phase space to the coset space $G|H$. $H$ is called the isotropy group of $|0\rangle$. Specializing to an Abelian U(1) group where $g(x,t) = e^{i\theta(x,t)}$, we find the action

$$S(\theta) = -\frac{1}{4\pi}\int d^2x[(\partial_x\theta)^2 + \partial_x\theta\,\partial_t\theta] \,. \tag{6}$$

The corresponding equation of motion is

$$\partial_x(\partial_x + \partial_t)\theta = 0 \tag{7}$$

and the solution is

$$\theta(x,t) = \theta_1(x-t) + \theta_2(t) \,. \tag{8}$$

The reader will certainly notice the similar form of this action with that in Floreanini and Jackiw's formulation for the chiral boson. There is, though, a crucial difference. The field $\theta$ here is a canonical scalar field while Floreanini and Jackiw used just one of its chiral components, being therefore nonlocal in nature, which clearly explains the curious commutation relation introduced by them in order to solve the model. Because of this difference the causality problems pointed out by Girotti *et al.*[18] in the Floreanini-Jackiw formulation do not apply here.

The Lagrangian $L_0$ defined by (6) is singular and has one second-class, primary constraint[15]

$$\Omega = \pi - \partial_x\theta \,, \tag{9}$$

where $\pi \equiv \delta L/\delta\dot{\theta}$, which does not commute with itself (at the Poisson-bracket level),

$$\{\Omega(x), \Omega(y)\} = -2\partial_x\delta(x-y) \,. \tag{10}$$

In order to convert this constraint to first class we use the algorithm proposed in Ref. 16. It is simple to see that the modified Lagrangian

$$L_0 \to L_1 = L_0 + \partial_x\theta_1\partial_t\theta_1 - 2\partial_t\theta_0\partial_x\theta_1 \tag{11}$$

will produce a pair of primary constraints,

$$\Omega \to \Omega_0 = \pi_0 - \partial_x\theta_0 + 2\partial_x\theta_1 \,, \qquad \Omega_1 = \pi_1 - \partial_x\theta_1 \,. \tag{12}$$

Here we have relabeled the original field $\theta \to \theta_0$, etc., and introduced a Wess-Zumino auxiliary field $\theta_1$. One can check that the matrix

$$C_{ij} = \{\Omega_i, \Omega_j\} \tag{13}$$

has $\det C = 0$, signaling the first-class nature of the set $\{\Omega_i,\ i=1,2\}$. The trouble now is that this set of constraints needs an extra secondary constraint to maintain the stability of the constraint hypersurface under time evolution, which is

$$\Omega_2 = \partial_x^2\theta_0 \,. \tag{14}$$

The solution is then to apply the same medication again. As before the introduction of a counterterm, along with an extra field $\theta_2$, will fix the problem with $\Omega_2$ but a new constraint will show up, and so on. After repeating this process $N$ times one finds the expression for the Lagrangian

$$L \to L_N = L_{N-1} + L_{\text{CT}}^{(N)}, \qquad (15)$$

where $L_{\text{CT}}^{(N)}$ is the level-$N$ counterterm given by

$$L_{\text{CT}}^{(N)} = \partial_t \theta_N \, \partial_x \theta_N - 2\partial_x \theta_N \sum_{k=0}^{N-1} (-1)^{k+N} \partial_t \theta_k . \qquad (16)$$

The total Wess-Zumino term up to this level is given by the sum of all counterterms,

$$L_{\text{WZ}}^{N} = \sum_{m=1}^{N} L_{\text{CT}}^{(m)} = \sum_{m=1}^{N} \left[ \partial_t \theta_m \, \partial_x \theta_m - 2\partial_x \theta_m \sum_{k=0}^{m-1} (-1)^{k+m} \partial_t \theta_k \right] . \qquad (17)$$

The system now has the following set of constraints:

$$\Omega_m = \pi_m - \partial_x \theta_m - 2 \sum_{k=m+1}^{N} (-1)^{k+m} \partial_x \theta_k, \quad m = 0, \ldots, N, \qquad (18)$$

$$\Omega_{N+1} = \partial_x^2 \theta_0 .$$

The last constraint $\Omega_{N+1}$ will always destroy the first-class nature of the set $\{\Omega_i, \, i = 0, \ldots, N\}$ and consequently one has no right to stop the loop process at any finite step. The Wess-Zumino term for the Abelian version of Sonnenschein's chiral boson is obtained from (17) by taking the limit $N \to \infty$. The equation of motion for the chiral boson $\theta_0$ becomes

$$\partial_x (\partial_x + \partial_t) \theta_0 = \partial_x \partial_t (\theta_1 + \cdots + \theta_N) , \qquad (19)$$

while all the Wess-Zumino auxiliary fields have identical equations, given by

$$\partial_x \partial_t (\theta_1 + \cdots + \theta_N) = \partial_x \partial_t \theta_0 . \qquad (20)$$

Observe that substituting (20) into (19) just gives us back the last constraint, which shows that only by making it disappear do we recover $\theta_0$ as the right-moving degree of freedom that we started with. This is achieved, as already mentioned, only with the introduction of an infinite number of counterterms.

Before concluding we would like to comment on the covariant quantization of our model, described by Eq. (17), which is presently under way.[19] To this end one can use the Batalin-Fradkin-Vilkovisky approach.[17] One then extends the phase space by incorporating the (infinite) constraints by means of infinite Lagrange multipliers and their conjugated momenta, as well as a set of infinite ghost fields of appropriate statistics. The construction of the extended action and the Becchi-Rouet-Stora-Tyutin (BRST) charge then follows the usual recipe given by the authors of Ref. 17, after a straightforward redefinition of the existing constraints in order to close the algebra of these objects. The analyses of the physical subspace generated by the application of the nilpotent BRST charge over the vacuum show the existence of only one chiral excitation, as one should expect.

In conclusion, we have derived the Wess-Zumino term for the U(1) Abelian version of Sonnenschein's chiral-boson model, making use of an iterative process which turns the original second-class constraint into a first-class one, with the introduction of appropriate counterterms. We found that it became necessary to introduce an infinite number of such counterterms to obtain the desired result. To include many chiral bosons into this scheme is straightforward. As discussed in the introduction, the gauge invariance of our chiral boson makes it a natural candidate to describe the edge excitations of quantum Hall liquids and should be of importance for the bosonic heterotic-string program. It would be interesting to examine if the coupling of this model to two-dimensional gravity will produce the same gravitational anomaly as that generated by Weyl fermions.

We thank Professor M. Stone for many useful discussions. This work is partially supported by NSF Grant No. PHY87-01775. The author is grateful to Coordenação de Aperfeiçoamento de Pessoal do Ensino Superior (CAPES), Brazil, for providing a fellowship which made this research possible.

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