Closed Timelike Curves Produced by Pairs of Moving Cosmic Strings: Exact Solutions

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Exact solutions of Einstein's field equations are presented for the general case of two moving straight cosmic strings that do not intersect. The solutions for parallel cosmic strings moving in opposite directions, each with $\gamma_s > (\sin 4\pi\mu)^{-1}$ in the laboratory frame show closed timelike curves (CTC's) that circle the two strings as they pass, allowing observers to visit their own past. Similar results occur for nonparallel strings, and for masses in (2+1)-dimensional spacetime. For finite string loops the possibility that black-hole formation may prevent the formation of CTC's is discussed.

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Traversable-wormhole solutions violating the averaged weak energy condition have been found[1] which exhibit closed timelike curves (CTC's). Observers traveling along CTC's can visit their own past creating a time machine with its associated causality paradoxes. It has generally been supposed that the laws of physics should be such as to prevent such solutions. Perhaps quantum field theory forbids violation of the averaged weak energy condition or perhaps our Universe was created without any traversable wormholes and the laws of physics prevent the topology changes that would accompany their construction.[2] Alternatively it may be that CTC's are allowed, providing that the laws of physics are augmented by a principle of self-consistency.[2] Thus it is quite important to see what kinds of solutions produce CTC's. The present paper presents exact solutions to Einstein's field equations for the general case of two moving straight cosmic strings that do not intersect. Remarkably, some of these solutions produce CTC's even though they (1) do not violate the weak energy condition, (2) have no singularities or event horizons, and (3) are not topologically multiply connected.

For straight cosmic strings,[3] the weak-field solution[4] was followed by an exact solution[5,6] whose exterior metric is given by

$$ds^2 = dr^2 + (1-4\mu)^2 r^2 d\phi^2 + dz^2 - dt^2,$$

where μ is the mass per unit length in geometrized units ($G=c=\hbar=1$), i.e., in Planck masses per Planck length, and whose associated exact interior solution for the uniform density case is given by

$$ds^2 = r_0^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dz^2 - dt^2,$$

where the density $\rho = 1/8\pi r_0^2$ and $P_z = -\rho$. The geometry of a $t=$const, $z=$const section of this solution is that of a cone with angle deficit $D = 8\pi\mu$ in the exterior (vacuum) region and that of a spherical cap in the interior region. The coordinates have ranges $-\infty < t < \infty$, $-\infty < z < \infty$, $0 \le \phi < 2\pi$, $0 \le \theta \le \cos^{-1}(1-4\mu)$, and $r_b \le r < \infty$, where $r_b = r_0[(1-4\mu)^{-2} - 1]^{1/2}$. Adopt a new coordinate $\phi' = (1-4\mu)\phi$. The exterior metric becomes

$$ds^2 = dr^2 + r^2 d\phi'^2 + dz^2 - dt^2, \quad (1)$$

where $0 \le \phi' < (1-4\mu)2\pi$ and $r \ge r_b$. Metric (1) is just the metric for Minkowski space in cylindrical coordinates where a wedge of angle deficit $D = 8\pi\mu$ is missing and points with coordinates $(r, \phi'=0, z, t)$ and $(r, \phi' = 2\pi - 8\pi\mu, z, t)$ are identified. (Interestingly, these solutions had been known[7] as mathematical solutions to the field equations without any physical interpretation as strings prior to the invention of cosmic strings.)

General static string solutions[5] can be found without cylindrical symmetry or uniform density. These have metrics of the form $ds^2 = ds'^2 + dz^2 - dt^2$, where ds'^2 is the metric of an arbitrary spacelike two-surface where the Gaussian curvature K can vary with position but is never negative and $\rho = -P_z = K/8\pi$.

The static solution for two parallel cosmic strings[5] separated by a distance $2d$ is constructed as follows. Adopt metric (1), replacing r and ϕ' by the Cartesian coordinates $x = r\sin(\phi' + 4\pi\mu)$ and $y = r\cos(\phi'+4\pi\mu) + d$. The metric is $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$, and applies provided $x^2 + (y-d)^2 \ge r_b^2$ and $|x| \ge (y-d) \times \tan(4\pi\mu)$ [points with $x = \pm(y-d)\tan(4\pi\mu)$ are identified] for $y > d + r_b\cos(4\pi\mu)$ (valid for $0 \le \mu \le \frac{1}{8}$). Consider now the three-surface $y=0$. It has a metric $ds^2 = dx^2 + dz^2 - dt^2$ and as part of a (3+1)-dimensional Minkowski space it has zero intrinsic and zero extrinsic curvature. Thus we can make a mirror-image second copy of the region $y \ge 0$ including its interior solution and join it back to back with the first region along the three-surface $y=0$ (see Fig. 1) (so that the second solution is the region $y \le 0$). The two copies obey all the matching conditions[8] along the surface $y=0$ because that surface in both solutions is a (2+1)-dimensional Minkowski space with zero intrinsic and extrinsic curvature.

Consider observers A and B at rest with respect to the cosmic strings whose world lines are given by $x_A(\tau) = x_0$, $y_A(\tau) = 0$, $z_A(\tau) = 0$, $t_A(\tau) = t$ and $x_B(\tau) = -x_0$, $y_B(\tau) = 0$, $z_B(\tau) = 0$, $t_B(\tau) = t$. As previously found,[5] observer

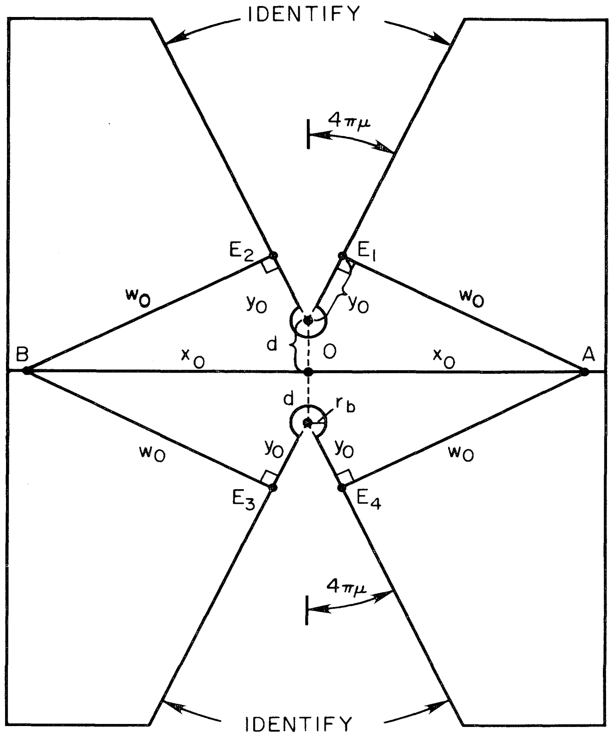

FIG. 1. Two-parallel-string static solution: (x,y) plane.

B sees three images of observer A. The central image is from a geodesic passing through the origin 0 (see Fig. 1), while the two outrigger images, displaced from the central image by an angle $\Delta\theta=4\pi\mu$ on each side, represent geodesics that pass through events E_1-E_2 and E_3-E_4. (Events E_1 and E_2 are identified as are E_3 and E_4.) Now $w_0^2=[x_0-y_0\sin(4\pi\mu)]^2+[d+y_0\cos(4\pi\mu)]^2$ and the value of y_0 picked to minimize w_0 is $y_0=x_0\times\sin(4\pi\mu)-d\cos(4\pi\mu)$ (see Fig. 1). Thus, $w_0<x_0$ if $d<y_0$, and the light beam going through E_1-E_2 arrives before the light beam going through 0 with a gravitational lensing time delay between the two images of $\Delta t=2(x_0-w_0)$. If a light beam passing through E_1-E_2 can beat the light beam passing through 0, then so can a rocket traveling at a high enough velocity $\beta_R<1$ relative to the string. Such a rocket can therefore connect two events in the $y=0$ $(2+1)$-dimensional Minkowski space which have a spacelike separation. Let the rocket begin at A at event $E_i=(x_0,0,0,-\beta_R^{-1}w_0)$ and end at B at event $E_f=(-x_0,0,0,\beta_R^{-1}w_0)$ (it takes the rocket a time $t=2\beta_R^{-1}w_0$ to traverse the path E_i to E_1-E_2 to E_f). The separation of E_i and E_f is spacelike providing that $x_0^2-\beta_R^{-2}w_0^2>0$, which can always be achieved for high enough $\beta_R<1$ since $w_0<x_0$. Now take the string solution for $y\geq 0$ and give it a velocity β_s in the $+x$ direction via a simple Lorentz transformation such that the events E_i and E_f become simultaneous in the laboratory frame: $\beta_s=w_0\beta_R^{-1}x_0^{-1}$. Take the solution for $y\leq 0$ and move it at velocity β_s in the $-x$ direction. The two solutions $y\geq 0$ and $y\leq 0$ can still be matched together[8] because the Lorentz transforms do not alter the fact that boundary surface $y=0$ in each solution is still a $(2+1)$-dimensional Minkowski space with zero intrinsic and zero extrinsic curvature. The rocket goes from E_i through E_1-E_2 and arrives at event E_f which is simultaneous with event E_i in the laboratory frame. Then by symmetry the rocket goes in the opposite direction past the oppositely moving string through E_3-E_4 and arrives back at event E_i which is simultaneous with event E_f in the laboratory frame. The rocket has now completed a CTC; it circles the two parallel cosmic strings as they pass each other in a sense opposite to that of the strings' relative motion.

Events E_i and E_f in the laboratory frame have coordinates $E_i=(\gamma_s^{-1}x_0,0,0,0)$ and $E_f=(-\gamma_s^{-1}x_0,0,0,0)$, where

$$\gamma_s^2=x_0^2/(x_0^2-w_0^2\beta_R^{-2})$$

since $\beta_R<1$

$$\gamma_s^2>x_0^2/(x_0^2-w_0^2)=x_0^2/(y_0^2-d^2)\,,$$

$$\gamma_s^2>\frac{[\sin(4\pi\mu)]^{-2}}{1-2d/x_0\tan(4\pi\mu)-d^2/x_0^2}\,,$$

where we can always choose $x_0\gg d$ so

$$\gamma_s>[\sin(4\pi\mu)]^{-1}\,.$$

For $\mu=10^{-6}$ expected for grand unified cosmic strings, $\gamma_s>8\times 10^4$ in order to produce CTC's.

In the original static solution (x,y,z,t):

$$E_1=(x_1,y_1,0,0),\quad E_2=(-x_1,y_1,0,0)\,,$$

$$E_3=(-x_1,-y_1,0,0),\quad E_4=(x_1,-y_1,0,0)\,,$$

where $x_1=y_0\sin(4\pi\mu)$ and $y_1=d+y_0\cos(4\pi\mu)$, but after the Lorentz transforms have been made in $y\geq 0$ and $y\leq 0$ regions, these events now have coordinates in the laboratory frame (x,y,z,t):

$$E_1=(\gamma_s x_1,y_1,0,\beta_s\gamma_s x_1)\,,$$

$$E_2=(-\gamma_s x_1,y_1,0,-\beta_s\gamma_s x_1)\,,$$

$$E_3=(-\gamma_s x_1,-y_1,0,\beta_s\gamma_s x_1)\,,$$

$$E_4=(\gamma_s x_1,-y_1,0,-\beta_s\gamma_s x_1)\,.$$

Again E_1-E_2 and E_3-E_4 are identified. It is possible to travel by rocket at speed β_r in the laboratory frame from E_2 to E_3 and from E_4 to E_1 if

$$\beta_r\beta_s\gamma_s y_0\sin(4\pi\mu)=d+y_0\cos(4\pi\mu)\,.$$

Since $\beta_r<1$ and we can always choose x_0 so $y_0\gg d$ we find

$$\beta_s\gamma_s>[\tan(4\pi\mu)]^{-1},\quad \gamma_s>[\sin(4\pi\mu)]^{-1}$$

as before. This represents a CTC where the rocket undergoes acceleration only at E_1-E_2 and E_3-E_4 (compared with the previous solution where the rocket undergoes acceleration only at E_i and E_f). In the laboratory frame it is clear how the CTC is created. The E_1-E_2 and E_3-E_4 identifications allow the particle to effectively travel backward in time twice in the laboratory frame. The identification of E_1-E_2 and E_3-E_4 is equivalent to having a complete Minkowski space without missing wedges where instantaneous (tachyon) travel in the string rest frames between E_1 and E_2, and E_3 and E_4, is possible. It is then perhaps not surprising that causality violations can be created.

If the dz^2 term in parallel string metrics is deleted,[5] we obtain solutions for masses[9] in (2+1)-dimensional spacetime. Thus the above solution with $z=0$ (deleting the dz^2 term in the metrics) constitutes the general solution of the noncolliding two-body problem in (2+1)-dimensional spacetime. Here mass is dimensionless, usually normalized to be the angle deficit. Thus CTC's are produced when two masses M pass each other with each having $\gamma > [\sin(M/2)]^{-1}$ in the center-of-mass frame.

Starting with the parallel static solution of Fig. 1, the general solution is obtained by applying to the $y \geq 0$ and $y \leq 0$ solutions all Lorentz boosts, rotations, and translations that map the three-surface $y=0$ into itself. Thus, the strings need not be parallel. Go to the parallel static solution of Fig. 1 and rotate the $y \geq 0$ and $y \leq 0$ solutions each by an angle ϕ in opposite directions in the (x,z) plane. The two solutions still match on the $y=0$ surface because it is still a three-surface with zero intrinsic and extrinsic curvature. Let world lines A and B in the combined solution have $x_A(\tau)=x_0$, $y_A(\tau)=0$, $z_A(\tau)=0$, $t_A(\tau)=t$ and $x_B(\tau)=-x_0$, $y_B(\tau)=0$, $z_A(\tau)=0$, $t_B(\tau)=t$. Then $E_1=(x_1\cos\phi, y_1, -x_1\sin\phi, 0)$ and $E_2=(-x_1\cos\phi, y_1, x_1\sin\phi, 0)$ so that $w_0^2=(x_0-x_1\cos\phi)^2+(x_1\sin\phi)^2+y_1^2$ and $y_0=x_0\sin(4\pi\mu)\cos\phi-d\cos(4\pi\mu)$ is the solution that minimizes w_0. Again $w_0^2-x_0^2=d^2-y_0^2$, so if we Lorentz boost the $y \geq 0$ solution by a velocity β_F in the $+x$ direction and the $y \leq 0$ solution by a velocity β_F in the $-x$ direction to produce CTC's, we require

$$\gamma_F^2 > \frac{[\sin(4\pi\mu)]^{-2}}{\cos^2\phi - 2d\cos\phi/x_0\tan(4\pi\mu) - d^2/x_0^2}$$

and since we can choose $x_0 \gg d$ we require $\gamma_F > [\sin(4\pi\mu)]^{-1}(\cos\phi)^{-1}$ in what follows. Now it is only the component of velocity perpendicular to the string which can be measured, and in the frame ($\beta_z = -\tan\phi/\gamma_F\beta_F > -1$) which equalizes and minimizes the measured string velocities we find that $\gamma_s=\gamma_F\cos\phi$, so $\gamma_s > [\sin(4\pi\mu)]^{-1}$ as before.

Since the production of the CTC's relies on nothing more than the gravitational lens effect, one might wonder whether they may be produced by simply firing masses at each other. Two masses separated by a distance $2d$ and observed from a distance x_0 will produce multiple images of a source at a distance x_0 behind them. Consider three of these images, central one and two outrigger ones separated from it by $\Delta\theta$. If y_0 is the impact parameter of an outrigger lensed ray, then the central image will be delayed relative to the other two providing $d < y_0$ so that it passes deeper into the potential well. One of the masses produces a total deflection $2\Delta\theta=4M/y_0$, where y_0 is the impact parameter. The time delay is of order $\Delta\theta^2 x_0$. To produce CTC's, in the center-of-mass frame we require $\gamma > (\Delta\theta)^{-1}$ so $2\gamma M > y_0 > d$. In this frame a total mass $2\gamma M$ is contained within $2d < 4\gamma M$, equal to the Schwarzschild radius, so rather than passing each other, the masses should be pulled together to form a black hole. This can prevent the CTC's. Cosmic strings can pass each other without effectively attracting each other. In a static configuration, $Q=M$ black holes do not attract, but at high speed, passing each other, they will attract and might also be expected to form a single black hole.

Can cosmic strings in nature ever pass each other with $\gamma_s > [\sin(4\pi\mu)]^{-1}$? Strings achieve high velocity in collapsing loops, and at kinks where $\gamma \rightarrow \infty$. An initially static circular loop remains circular as it collapses. Its total mass $M_s=\mu\int\gamma_s\,dl$ is a constant of the motion[10] (ignoring gravitational radiation). So $\gamma \rightarrow \infty$ as its circumference goes to zero. Perturb this solution slightly (making the loop elliptical) and it collapses to a double line.[10] A slight additional perturbation (with higher-frequency terms) creates a loop which does not self-intersect.[11] Thus it should be possible to find closed-loop solutions where nearly straight segments pass each other at high speed. Hawking[12] has produced an argument that if cosmic censorship is valid, a circular loop collapsing with $\gamma=\infty$ (approximating the final phases where $\gamma \rightarrow \infty$) will produce a black hole with a loss of at most a fraction $1-2^{-1/2}=29.3\%$ of its mass energy in the form of gravitational radiation. Thus, while it is still outside the horizon, $M_s \geq \mu\gamma_s 4\pi M_{\mathrm{BH}} > 4\pi\mu\gamma_s 2^{-1/2}M_s$ so $\gamma_s < 2^{1/2}(4\pi\mu)^{-1}$. This offers some hope that black-hole formation may in practice prevent γ_s from exceeding $[\sin(4\pi\mu)]^{-1}$. Thorne's[13] conjecture that black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is less than $4\pi M$ would imply $\gamma_s < (4\pi\mu)^{-1}$ for the circular loop case. But we are interested in the case of two long passing strings. As the strings pass each other, a thin envelope of length L and width $2d$ [maximum circumference $\sim 2(L^2+4d^2)^{1/2}$] can be wrapped around them so as to contain a mass $M \sim 2L\gamma_s\mu$. So for $L \gg d$ and $\gamma_s > (4\pi\mu)^{-1}$, Thorne's conjecture appears to be violated by our exact solution which does not have any event horizons. If Thorne's conjecture were true for finite-length string segments that were part of loops, it could conveniently prevent the CTC's, but the exact solutions presented here warn that Thorne's conjecture may be violated. Kinks from separate loops may pass each other at high speed (but they are not approximately straight). An advanced civilization could always in principle accelerate separate loops to high speed by towing them gravitationally with very massive rockets. Alternately, there may even be extremely rare segments of pairs of infinite strings that pass at high speed. Can black-hole formation always prevent the formation of CTC's, or hide them behind event horizons? In (2+1)-dimensional spacetime we do not have the luxury of a possible black-hole escape clause. There, CTC's can be eliminated only by restricting the initial conditions or by disallowing massive bodies.

We are left with a number of questions. Is it possible that with the weak energy condition (WEC) CTC's visible from infinity can be produced with sources that extend to infinity (like the exact string solutions above and Tipler's rapidly rotating infinite cylinder[14]), but for finite sources with realistic initial conditions any CTC's are hidden behind event horizons[15] [as in the analytic continuation of the Kerr-Newman $a^2+e^2 \leq m^2$ ($a \neq 0$) metric[16]], unless the averaged WEC is violated as in the wormhole[1] solutions? Or could there ever be finite, nonsingular WEC sources that produce causality violating regions extending to infinity like the above exact string solutions and the Kerr-Newman $a^2+e^2 > m^2$ ($a \neq 0$) naked ring singularity?[16,17] Normally we wish to prevent solutions with causal paradoxes. Linde[18] has proposed that chaotic inflation can produce universes with different macroscopic dimensionalities of spacetime. Do we wish to disallow (2+1)-dimensional spacetimes because they can produce CTC's in specific cases? Do we wish to disallow cosmic strings because they can in principle produce CTC's or can we in practice always rely on black-hole formation to prevent CTC's? Alternatively, should the existence of these solutions push us to accommodate CTC's using a principle of self-consistency? The solutions presented here should be useful in future causality studies.

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