Reversibility and Noise Sensitivity of Josephson Arrays

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We predict that series arrays of point-contact Josephson junctions, shunted by a resistive load, should be unusually sensitive to noise. This behavior stems from a dynamical symmetry which prohibits the existence of in-phase attractors. Simulations verify that breaking the symmetry can radically improve the phase-locking performance of the array.

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Josephson-junction arrays hold great promise for future technologies, including high-frequency voltage sources, voltage standards, parametric amplifiers, and generators of squeezed states in quantum electronics.[1] At the same time, these arrays are of fundamental interest as examples of nonlinear dynamical systems with many degrees of freedom. Recent work has shown that these two perspectives are not disjoint: An understanding of the underlying structure of the circuit equations can give insights into the performance of various arrays.[2-7]

Viewed as a dynamical system, each junction is a nonlinear oscillator. As a practical matter, very large arrays are desirable because individual junctions generate relatively little power. Consequently, researchers are interested in large arrays: If the junctions oscillate in perfect synchrony—so-called in-phase operation—the total power generated by an array of N junctions is expected to scale as N^2. Thus, the fundamental problem is to understand the phase-locking properties of N coupled oscillators, i.e., the conditions under which the in-phase state is an attractor for the dynamics.

Hadley and co-workers have mapped out stability diagrams of the in-phase state for a variety of array configurations, based largely on numerical integration of the circuit equations.[3] More recently, more sophisticated bifurcation analyses have been employed to get improved predictions for certain geometries.[6]

In this Letter we show that certain Josephson arrays possess an important dynamical symmetry which has far-reaching effects on the observed dynamics. Curiously, despite the presence of dissipative elements, the governing equations obey a kind of time-reversal invariance, which prohibits the existence of in-phase attractors. This, in turn, has fundamental consequences for the performance of the arrays, in particular, as regards the robustness of the in-phase operation at finite temperatures. On the other hand, altering the coupling load can remove the symmetry, leading to dramatically improved performance.

Consider a linear array of N Josephson junctions (see Fig. 1). The array is driven by a constant bias current I_B; the junctions are coupled via a parallel load resistor R. The governing dynamical equations are

$$(\hbar/2er)\phi_k' + I_c \sin\phi_k + I = I_B\,, \tag{1}$$

$$RI = V(t) = \frac{\hbar}{2e}\sum_j \phi_j'\,, \tag{2a}$$

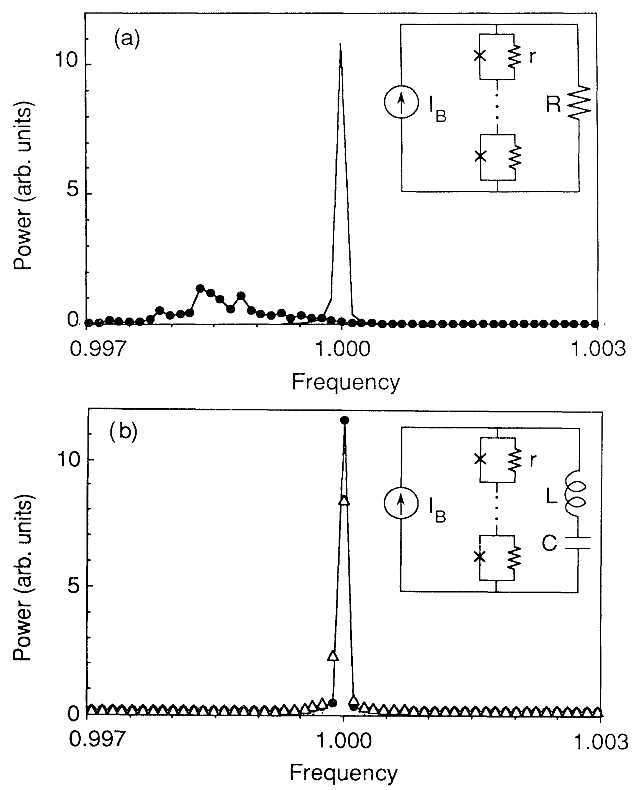

FIG. 1. (a) Power spectrum of the total voltage across an array with a pure resistive load, from simulation of Eqs. (1) and (2a), with $N=5$, $R=12.5r$, and $I_B=3I_c$. Frequencies are scaled by the fundamental frequency of the noise-free system. The solid line is for noise intensity $\kappa=0$; the circles are for $\kappa=5\times10^{-5}\hbar I_c/2er$. The length of each time series is about 8400 oscillation periods. Inset: Schematic of circuit. (b) Power spectrum with LC load, from simulation of Eqs. (1) and (2b), with $N=5$, $L=5\hbar/2eI_c$, $C=0.1\hbar/2eI_cr^2$, $I_B=2.143I_c$, and $\kappa=0$ (solid line), $\kappa=5\times10^{-5}\hbar I_c/2er$ (circles), $\kappa=5\times10^{-3}\hbar I_c/2er$ (triangles). Inset: Schematic of circuit.

where $k=1,\ldots,N$. Here, I is the current passing through the load, ϕ_k is the phase difference of the macroscopic wave function across the kth junction, r is the junction resistance, I_c is the critical current, e is the electron charge, $\hbar$ is Planck's constant divided by 2π, and the prime denotes differentiation with respect to time. In Eq. (1), which follows from current conservation, we have assumed that the capacitance of the Josephson junctions is negligible, which is appropriate for a variety of junction types, e.g., superconductor–normal-metal–superconductor junctions.[8] Equation (2a) states that the voltage across the load resistor is equal to the voltage $V(t)$ across the entire array. Note the crucial role played by the load: In its absence, the junctions are completely uncoupled; only in its presence will the voltage oscillations of the individual elements interact. Ideally, one wants to find operating conditions under which the coupled array settles into the in-phase state, so that $\phi_i(t)=\phi_1(t)$, for all i.

Simulations of the dynamical behavior of this array are revealing. Figure 1(a) shows the power spectrum of the total voltage across the array, $V(t)$, obtained by numerical integration of the governing equations, both with and without the influence of external noise. The system is initially put in the in-phase state, i.e., $\phi_1=\phi_2=\cdots=\phi_N$ at $t=0$. As expected, in the absence of noise, the individual oscillations add coherently, leading to a sharp line at the fundamental frequency. However, the presence of weak noise sources leads to a substantially degraded output. Specifically, the noise was modeled by adding a random term $\sqrt{\kappa}\xi_k(t)$ to Eq. (1), representing the noise current due to the individual junction resistances. The $\xi_k(t)$ are independent δ-correlated random functions with zero mean and unit variance; κ is proportional to the power of each noise source.

Note that this noise sensitivity is not inevitable. In Fig. 1(b), the resistor R is replaced by a different load, namely, an inductor and capacitor in series. The parameters have been chosen so that the resulting power spectrum of the noise-free system is nearly identical to the case of the pure resistive load.[9] Despite the similarity of the noise-free behavior, we see that now the dynamics are far more robust in the presence of noise—in fact, even with an input noise power 100 times greater than that used in Fig. 1(a), the spectral line in Fig. 1(b) is hardly broadened at all. (Note that the improved performance is not due to resonance: The LC resonant frequency is well separated from the observed frequency.[9])

We now show that the behavior displayed in Fig. 1 can be traced directly to the structure of the dynamical equations governing the arrays. In particular, we note that Eq. (1) is invariant under a kind of time reversibility, namely, the transformation $(\phi_k,t)\to(\pi-\phi_k,-t)$; Eq. (2a) also has this symmetry. (Recall that ϕ_k is a phase, defined modulo 2π.) Note that this symmetry is *not* respected by the LC load, for which the voltage equation (2a) is replaced by

$$LI'+\left(\frac{1}{C}\right)\int_0^t I(s)\,ds=V(t)=\frac{\hbar}{2e}\sum_j\phi_j'\,, \tag{2b}$$

where L and C are the load inductance and capacitance, respectively.

The existence of reversibility has dramatic consequences for the observed dynamics. As we now show, it follows that *any such system has no in-phase periodic attractor.* Suppose first that there is an asymptotically stable in-phase periodic orbit, denoted by Γ. Then, any initial condition $\{\phi_k\}$ sufficiently close to Γ must approach Γ as $t\to\infty$. On the other hand, the orbit starting from the symmetry-related point $\{\pi-\phi_k\}$ must approach the symmetry-related version of Γ, but as $t\to-\infty$. But the symmetry-related version of Γ is Γ itself. So, viewed in forward time, we have the situation that for every trajectory attracted to Γ, there is another trajectory which is repelled from Γ. It follows that Γ is not an attractor.

Though simple, this observation is a very strong one. For example, it was previously shown that the in-phase state of the circuit of Fig. 1(a) was linearly neutrally stable.[10] This was based on a linear stability analysis; however, such techniques are not sufficiently powerful to determine the nonlinear effects, which in this case dominate (since the linear terms vanish). We see now that the in-phase state is not attracting; moreover, any modification of the array which retains this symmetry is likewise prohibited from having an in-phase attractor.

The neutral stability of the in-phase orbit causes the system to be unusually sensitive to noise, and results in the broad spectral line in Fig. 1(a). We now present a simple heuristic approach to finding the scaling of the power spectrum, as a function of the noise strength κ and the array size N. A rigorous calculation would require the study of a large system of nonlinear, coupled stochastic differential equations. However, we believe the following calculation correctly captures the essential phase-space geometry of the full problem, and so should yield sensible results.

We now estimate the power spectrum. The allowed values of the ϕ_k define an N-torus; the in-phase orbit lies along the diagonal $\phi_1=\phi_2=\cdots=\phi_N$. For a pure resistive load, the random perturbations cause the orbit to diffuse over the entire N-torus. For low noise levels, this diffusion is very slow; nevertheless, in the long-time limit, the system freely diffuses over the whole torus.

Based on this picture, we can estimate the expected power spectrum. The voltage across the kth junction is proportional to ϕ_k', which we approximate by

$$\phi_k'=\omega+a\cos[\omega t+\alpha_k(t)]\,,$$

where ω is the oscillation frequency, and a is the amplitude at the fundamental frequency; higher harmonics are ignored here. Because of the slow diffusion, the relative phases $\alpha_k(t)$ are taken to be independent continuous-time random walks. These variables $\alpha_k(t)$ may be considered phase-space variables on an N-torus. In order to capture the essential feature of neutral stability in the simplest possible way, we take Langevin equations $\alpha_k' = \xi_k(t)$, where the ξ_k represent independent white-noise processes with zero mean and variance κ. Taking initial conditions that all the $\alpha_k(0)=0$, the joint probability density is therefore

$$P(\boldsymbol{\alpha}(t)) = \prod_k (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{\alpha_k}{2\sigma^2}\right),$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_N)$, and $\sigma^2=2\kappa t$. It is a straightforward matter to compute the autocorrelation function $\langle V(t)V(t+\tau)\rangle$ and the corresponding power spectrum $S(\Omega)$, with the result

$$S(\Omega) = Na^2\left(\frac{\kappa}{\kappa^2+(\Omega-\omega)^2}+\frac{\kappa}{\kappa^2+(\Omega+\omega)^2}\right),$$

where we have suppressed the δ-function contribution at $\Omega=0$.

A similar calculation can be performed for the LC load, in which case it is possible to have an in-phase attractor, according to numerical simulations.[2] The above phase-space picture is now modified, so that the ensemble of phases freely diffuses *along* the limit cycle, but is attracted to it in $N-1$ perpendicular directions. The simplest Langevin equations of this type are $\alpha_1' = \xi_1(t)$, $\alpha_k' = -\lambda\alpha_k + \xi_k(t)$, $k>1$, which imply that the probability density has the same form as before, but with widths given by

$$\sigma_1^2 = 2\kappa t,$$

$$\sigma_k^2 = (\kappa/2\lambda)[1-\exp(-2\lambda t)], \quad k=2,\ldots,N.$$

Here, λ is the Lyapunov exponent of the limit cycle; physically, λ measures the strength of the attraction in the neighborhood of the in-phase state. In an ensemble picture, one imagines that the probability density is confined to a band of width $\kappa/2\lambda$ in each of $N-1$ directions orthogonal to the attractor, centered on the in-phase state; meanwhile, the system is free to diffuse along the limit cycle. Though the calculation of the power spectrum is somewhat lengthy, the dominant result is easy to understand: Instead of a broadened line, there is now a δ function at $\Omega=\omega$, in addition to a small broadband contribution:

$$S(\Omega) = a^2N(N-1)\exp(-\kappa/\lambda)\delta(\Omega-\omega) + \text{broadband}.$$

Comparison of the power spectra for these two cases reveals two important points. First, the total (integrated) power scales as N^2 for the LC load, but only linearly with N for the R load. In effect, the absence of an in-phase attractor results in incoherent superposition of the oscillations of the individual elements, while the presence of a limit cycle leads to coherent superposition. Note that the coherence is not quite perfect [scaling as $N(N-1)$ rather than N^2] owing to the neutral stability of the dynamics along the in-phase orbit, which is a general feature of limit cycles in autonomous dynamical systems. Naturally, this imperfect coherence is negligible for large arrays. The second major result is that the width of the line for the R load is proportional to the noise strength κ, while for the LC load the line remains sharp, though somewhat smaller in magnitude. These characteristics are consistent with the spectra of Fig. 1.

Since the above effects are the result of an underlying symmetry, it is important to consider what happens if the symmetry is only slightly broken. For example, we have considered the case where the junction capacitance is negligible, and this is not always the case, for example, in large-area tunnel junctions. Dynamically, this results in the presence of a second derivative in Eq. (1), which breaks the reversibility symmetry, and may render the in-phase orbit asymptotically stable. This is consistent with previous numerical results,[3,10] which showed that in-phase operation was stable for nonzero capacitance, but ever more weakly so as the capacitance decreased to zero. On the other hand, the results described above are valid as long as the symmetry is intact, even for substantially different circuit configurations, including two-dimensional series arrays.

A second important question involves whether the noise sensitivity displayed in Fig. 1(a) persists for arrays consisting of nonidentical junctions, an issue of obvious practical importance. This is a difficult problem, since the locked state no longer corresponds to the simple case where $\phi_j=\phi_k$ for all j and k. Physically, however, we expect that making the junctions different would reduce their tendency to phase lock, leading to an even worse degradation of the spectral line in the presence of noise. Indeed, numerical simulations verify this picture. At the same time, the robustness against noise displayed in Fig. 1(b) should also persist for arrays of nonidentical junctions, since the sharpness of the spectral line is specifically due to the presence of an attracting orbit, and not the existence of underlying symmetries. In this case, however, while the line remains sharp, we might expect its amplitude to diminish as a spread in junction parameters is introduced.

Finally, we note that the presence of a symmetry involving time reversal is somewhat unfamiliar considering the dissipative nature of the physical system. In fact, we have seen that manifestly *diminishing* the dissipation, by replacing the load resistor by nondissipative elements, can destroy the dynamical reversibility. We emphasize that analogous time reversibility has been reported previously by researchers studying problems involving dissipative dynamical systems. In particular, such reversibility arises in a laser system studied by Politi, Oppo, and

Badii,[11] and in the study of particle sedimentation in a highly viscous fluid.[12,13] As we have found in the Josephson-junction array, the presence of reversibility has important consequences for the observed dynamics in those problems as well.

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