Measurement-Induced Oscillations of a Highly Squeezed State between Super- and Sub-Poissonian Photon Statistics

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A general quantum theory of continuous photodetection is applied to an initially squeezed state to clarify effects of quantum-mechanical measurement backaction on a highly nonclassical field. The remaining photon field is found to oscillate in time between super- and sub-Poissonian photon statistics due to the backaction of the photon-number measurement.

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Recently, a general quantum theory of *continuous* photodetection process has been developed that describes a *nonunitary* state evolution of a measured photon field, and the theory has been applied to explore time evolutions of typical quantum states.[1-6] This has shown that state reduction depends strongly on both initial photon statistics and readout information (photoelectron statistics). The physical origin of the change in photon statistics brought about by measurements is identified as the renormalization of the density operator associated with the vanishing probability of a vacuum state. In this Letter, we apply this general theory to an initial state that is highly squeezed; in particular, we report novel measurement-induced oscillations of the initially highly squeezed state between super- and sub-Poissonian photon statistics. Measurement backaction on the observed system is shown to play an essential role in yielding such a novel phenomenon.

We consider photon counting for a single-mode field inside a cavity and discuss the time evolution of the density operator of the field *remaining* inside the cavity during photon counting. According to measurement, we calculate[7] the photon statistics remaining inside the cavity at time t. In particular, we show that if one photon is detected instantly at time t, then the conditioned statistical moments of the remaining photon field undergo various oscillations depending upon the initial quantum state of the field.

Let us briefly review the time evolution of a remaining photon field where we read out all information concerning registrations of photocounts in *real time* throughout the measurement period. Such a process is referred to as the quantum photodetection process of forward recurrence times (QPF).[1] The continuous photodetection process consists of two elementary processes, i.e., one-count and no-count processes described by the superoperators J and S_τ, respectively.[1-6] Suppose that a measurement process started at $t=0$ and ended at $t=T$, and that m photons were registered at times τ_j $(j=1, 2, \ldots, m)$ with no further photons registered during the measurement period. Then the density operator of the photon field, $\rho_m^{\rm QPF}(\tau_1,\tau_2,\ldots,\tau_m;0,T)$, immediately after the measurement process is given by[1-3]

$$\rho_m^{\rm QPF}(\tau_1,\tau_2,\ldots,\tau_m;0,T)=\frac{S_{T-\tau_m}JS_{\tau_m-\tau_{m-1}}J\cdots S_{\tau_1}\rho(0)}{\mathrm{Tr}[S_{T-\tau_m}JS_{\tau_m-\tau_{m-1}}J\cdots S_{\tau_1}\rho(0)]} \tag{1a}$$

$$=\frac{\exp[-(i\omega+\frac{1}{2}\lambda)\hat a^\dagger\hat a T]\hat a^m\rho(0)(\hat a^\dagger)^m\exp[(i\omega-\frac{1}{2}\lambda)\hat a^\dagger\hat a T]}{\mathrm{Tr}[\rho(0)(\hat a^\dagger)^m\exp(-\lambda\hat a^\dagger\hat a T)\hat a^m]}, \tag{1b}$$

where $\rho(0)$ is the initial density operator of the photon field, $\hat a$ $(\hat a^\dagger)$ is the creation (annihilation) operator of the relevant mode of the photon field, and λ represents the probability of one photoelectron being registered per unit time per one photon. The two superoperators are defined as $J\rho(t)\equiv\lambda\hat a\rho(t)\hat a^\dagger$ and $S_\tau\rho(t)\equiv\exp[-(i\omega+\frac{1}{2}\lambda)\hat a^\dagger\hat a\tau]\rho(t)\times\exp[(i\omega-\frac{1}{2}\lambda)\hat a^\dagger\hat a\tau]$. Because the right-hand side of Eq. (1) no longer depends on τ_j $(j=1,2,\ldots,m)$, we will henceforth denote the quantity $\rho_m^{\rm QPF}(\tau_1,\tau_2,\ldots,\tau_m;0,T)$ simply as $\rho_m^{\rm QPF}(T)$.

The denominator of Eq. (1a) is denoted as $P_m^{(\rm forward)}(\tau_1,\tau_2,\ldots,\tau_m;0,T)$ which is called the probability distribution of forward recurrence times (PDF).[8] The kth-order photon-number moments, $\langle n^k(t^+)\rangle=\mathrm{Tr}[\rho_m^{\rm QPF}(t)(\hat a^\dagger\hat a)^k]$ $(k=1, 2,\ldots)$, immediately after the QPF can be expressed in terms of the PDF as[3]

$$\langle n^k(t)\rangle=\sum_{j=1}^{k}\frac{1}{\lambda^j}\frac{P_{m+j}^{(\rm forward)}(\tau_1,\tau_2,\ldots,\tau_m,\overbrace{t,\ldots,t}^{j\text{ times}};0,t)}{P_m^{(\rm forward)}(\tau_1,\tau_2,\ldots,\tau_m;0,t)}\sum_{l=1}^{j}\frac{(-1)^{j+l}l^k}{l!(j-l)!}. \tag{2}$$

Here we introduce the Fano factor $F(t)$ which is defined as the ratio of the photon-number variance, $\langle[\Delta n(t)]^2\rangle\equiv\langle n^2(t)\rangle-\langle n(t)\rangle^2$, to the average photon number, $\langle n(t)\rangle$, i.e., $F(t)\equiv\langle[\Delta n(t)]^2\rangle/\langle n(t)\rangle$. This factor takes values greater than unity for super-Poissonian states (e.g., thermal state), less than unity for sub-Poissonian states, or equal to unity for Poissonian states (e.g., coherent state).

We now apply this theory of continuous photodetection process to a quadrature-amplitude-squeezed state, which is generated from a coherent state $|\alpha\rangle$ via a unitary transformation,[9] i.e., $|\alpha,r\rangle \equiv S(r)D(\alpha)|0\rangle$. Here $S(r) \equiv \exp\{r[\hat{a}^2-(\hat{a}^\dagger)^2]/2\}$, $D(\alpha)\equiv\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})$, and $|0\rangle$ are the squeezing operator, the displacement operator, and the vacuum state vector, respectively. The initial density operator of this squeezed state is then given by $\rho(0)=|\alpha,r\rangle\langle\alpha,r|$. We assume for simplicity that α is real and the squeezing parameter is non-negative, $r \geq 0$. Then, the density operator of the initially squeezed state in the QPF is given as[3]

$$\rho_m^{\mathrm{QPF}}(t)=\frac{1}{N_m(t)}\sum_{\mu,\nu=0}^{\infty}\frac{1}{\sqrt{\mu!\nu!}}\left[\frac{\tanh r}{2}\right]^{(\mu+\nu)/2}\exp(-i\Omega\mu t+i\Omega^*\nu t)H_{\mu+m}\left[\frac{\alpha}{\sqrt{\sinh 2r}}\right]H^*_{\nu+m}\left[\frac{\alpha}{\sqrt{\sinh 2r}}\right]|\mu\rangle\langle\nu|\,, \tag{3}$$

where $\Omega=\omega-\frac{1}{2}i\lambda$, $N_m(t)$ is a normalization constant, and $H_n(z)$ are the Hermite polynomials, i.e.,

$$N_m(t)\equiv 2^m\left\{\frac{\partial^m}{\partial y^m}\left[\frac{1}{(1-y^2)^{1/2}}\exp\left[\frac{2y}{1+y}\frac{\alpha^2}{\sinh 2r}\right]\right]\right\}_{y=e^{-\lambda t}\tanh r}, \tag{4a}$$

$$H_n(z)\equiv(-1)^n\exp(z^2)\frac{d^n}{dz^n}\exp(-z^2)\,. \tag{4b}$$

Hence the PDF is[3]

$$P_m^{(\mathrm{forward})}(\tau_1,\tau_2,\ldots,\tau_m;0,T)=N_m(T)\frac{(\lambda\tanh r)^m}{\cosh r}\exp\left[-\lambda\sum_{j=1}^{m}\tau_j+\alpha^2(\tanh r-1)\right]. \tag{5}$$

Now let us explore the time evolution of a highly squeezed state (i.e., for large r) in the QPF. Figures 1(a)–1(c) show the time development of the average photon number, the photon-number variance, and the Fano factor, respectively, in the QPF, calculated using Eqs. (2) and (5). It is assumed that photons were detected at $\tau_1,\ldots,\tau_4$. From the figures we find that in the no-count process both the average photon number and the photon-number variance decrease monotonically in time, but that the photon-number variance decreases more rapidly than the average photon number. Hence, the Fano factor decreases monotonically in this process. On the other hand, in the one-count process the average photon number decreases, whereas the photon-number variance increases. Hence, the Fano factor increases abruptly in this process. Therefore, the time development of the Fano factor shows oscillations as time progresses. In general, the Fano factor decreases monotonically from $F>1$ to $F<1$ in the no-count process, while it increases abruptly from $F<1$ to $F>1$ in the one-count process. This phenomenon may therefore be referred to as *"measurement-induced Fano-factor oscillations."* Such novel oscillations generally occur when the following two conditions are satisfied: (i) the squeezing parameter is large (large $r>0$) and (ii) the coherent component is large (large $|\alpha|$). No Fano-factor oscillations around unity have been found when r is small.[3] After a sufficiently long time, $F(t)$ remains less than unity even in the one-count process, although it approaches unity from below.

The underlying physics of the Fano-factor oscillations can be explained as follows. The no-count process may occur in two different situations: when the premeasurement state was the vacuum state or when it had an anti-

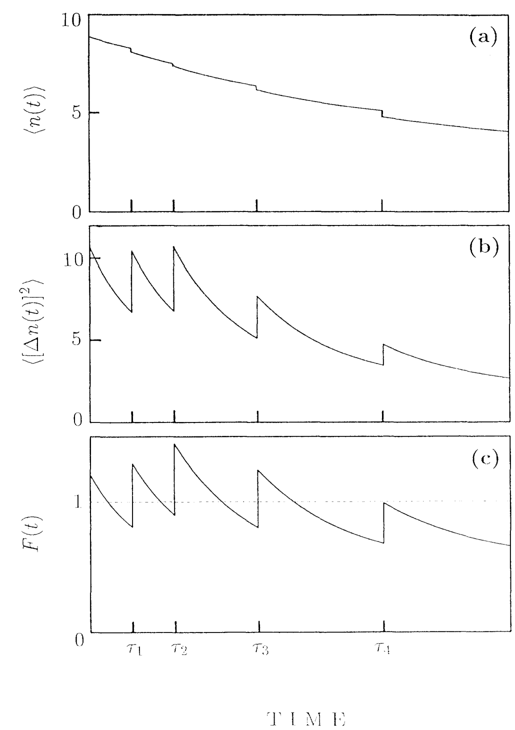

FIG. 1. Temporal developments of (a) the average photon number $\langle n(t)\rangle$, (b) photon-number variance $\langle[\Delta n(t)]^2\rangle$, and (c) the Fano factor $F(t)$ in the QPF, where $\lambda t\in[0,1]$. The initial state is chosen to be the quadrature-amplitude-squeezed state with $\alpha=8.0$ and $r=1.1$. One-count processes are assumed to occur at $\tau_1,\ldots,\tau_4$. Measurement-induced Fano-factor oscillations are clearly seen in (c).

bunched property. Since the probability of the vacuum state is very small for a highly squeezed state with a large coherent component, the first situation is unlikely. The more likely situation is that the measured field is antibunched. Since antibunching leads to a sub-Poissonian property within a time shorter than a certain characteristic time,[10,11] we can conclude that the measured field is sub-Poissonian. Thus the Fano factor decreases from $F>1$ to $F<1$. (Note that a highly squeezed state exhibits a super-Poissonian character.) As time passes in the no-count process, however, the contribution of the vacuum component becomes more and more significant until it overwhelms the second situation. The Fano factor thus starts to increase even in the no-count process and approaches unity. In fact, when only no-count processes occur, the Fano factor first decreases from above to below unity, and then it gradually increases towards unity from below, as shown in Fig. 2.

On the other hand, the one-count process modifies the matrix elements of the density operator as[3]

$$\rho_{mn}(t^+)=\frac{\sqrt{(m+1)(n+1)}}{\langle n(t)\rangle}\rho_{m+1,n+1}(t)\,. \tag{6}$$

The matrix elements with large m and n [such that $(m+1)(n+1)>\langle n(t)\rangle^2$] are enhanced by the one-count process. This results in an enlargement of the photon-number variance; that is, the tail part of the photon-number distribution function is enhanced to enlarge the photon-number variance and to yield a jump in the Fano factor to above unity, particularly in the highly squeezed state.

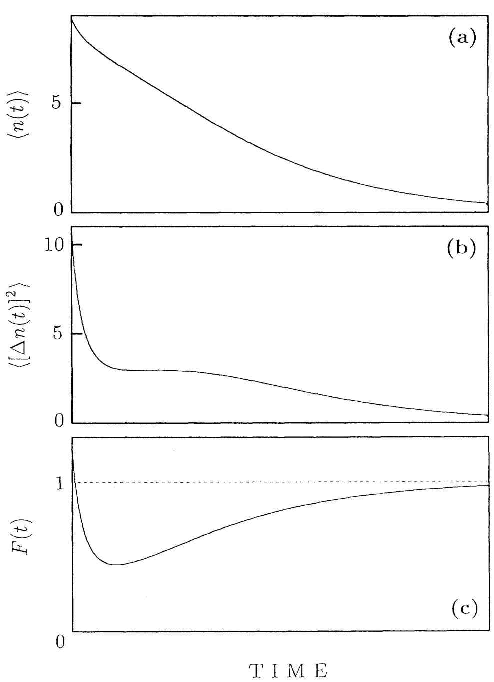

FIG. 2. Temporal developments of (a) the average photon number $\langle n(t)\rangle$, (b) the photon-number variance $\langle[\Delta n(t)]^2\rangle$, and (c) the Fano factor $F(t)$ during the no-count process, where $\lambda t\in[0,40]$. The initial state is chosen to be the same as in Fig. 1.

We recall the fact that whether the Fano factor increases or decreases in the one- or no-count process depends on the third cumulant of the photon number, $\langle[\Delta n(t)]^3\rangle$; that is,[3]

$$F(t^+)=\frac{\langle[\Delta n(t)]^2\rangle-[F(t)]^2+\langle[\Delta n(t)]^3\rangle/\langle n(t)\rangle}{\langle n(t)\rangle+F(t)-1}\,, \tag{7a}$$

$$F(t+\tau)=F(t)-\lambda\int_t^{t+\tau}\left\{[F(t')]^2-\frac{\langle[\Delta n(t')]^3\rangle}{\langle n(t')\rangle}\right\}dt'\,, \tag{7b}$$

for one- and no-count processes, respectively. In classical states, the Fano factor never becomes less than unity. Therefore, we can observe that the Fano-factor decrease *across unity* occurs only in a nonclassical state. Moreover, the third cumulant of the photon number (which is equivalent to the skewness of the photon distribution) of the classical states always satisfies $\langle[\Delta n(t)]^3\rangle\geq\langle[\Delta n(t)]^2\rangle^2/\langle n(t)\rangle$. Then the Fano factor must decrease in the no-count process because the integrand of Eq. (7b) is negative. This inequality does not always hold for nonclassical states. When this inequality is reversed $F(t)$ increases in the no-count process, as shown in Fig. 2(c).

As an example, we shall compare this case with the classical case of an initially thermal state. Here, the Fano factor decreases monotonically towards unity from $F(0)=1+\langle n(0)\rangle$ [where $\langle n(0)\rangle$ is the initial average photon number] as[3]

$$F_{\text{thermal}}(t)=\frac{1+\langle n(0)\rangle}{1+\langle n(0)\rangle(1-e^{-\lambda t})}>1\,; \tag{8}$$

that is, the initially thermal state maintains its super-Poissonian character throughout the measurement period. There are neither discontinuous changes nor oscillations in the Fano factor for an initially thermal state. Even if we choose the parameters α and r of a quadrature-amplitude-squeezed state to satisfy $F(0)=1+\langle n(0)\rangle$, the photon statistics of an initially squeezed state evolve differently from those of an initially thermal state. Moreover, in the case of an initially coherent state, the Fano factor remains unchanged in the QPF to take a constant value, $F(t)\equiv 1$ for $\forall t\geq 0$. Therefore, the novel evolution of the Fano factor arises from highly nonclassical properties of photons and is, therefore, unique to a *highly* amplitude-squeezed state.

Finally, some comments are appropriate here. First, we should note that all that we have calculated are ensemble-averaged quantities, $\langle n(t)\rangle$, $\langle[\Delta n(t)]^2\rangle$, and $F(t)$, which can be obtained not from the result of a single measurement but from the results of many repeated measurements. This can be seen from the fact that they are calculated from the density operator which is conditioned on the result of a single measurement. We start with an initial density operator which means that we prepare a large ensemble of quantum states with the same statistical properties. Out of this large ensemble we select a particular set of the quantum states according to the result of a single measurement. Even after the single measurement, we can still discuss the statistical properties of the field, which can be verified by performing many repeated measurements over the selected subensemble. This procedure is essential in a continuous measurement scheme.

Second, we stress again the importance of measurement backaction on the measured photon field. This plays a key role in our analysis for a fully quantum-mechanical measurement-theoretical photodetection scheme. If we neglect the effects of the measurement backaction, the Fano-factor oscillation disappears. In fact, if we set $\lambda = 0$ in our formulas, then they reduce to the results previously obtained without regard to measurement backaction. Thus, novel Fano-factor oscillations depend essentially on the measurement backaction, which has never been considered before in a unified manner for a highly squeezed state in photodetection processes.

Third, it is important to notice that we discuss the density operator of the *remaining* field inside a cavity during the photodetection process. This is quite different from the treatment by other authors[12] who calculated the average number of photocounts *registered* in a time interval. In fact, it has already been widely accepted that the average photon number of the remaining field decreases in time during a photodetection process if we take the effects of the measurement backaction into account.[1-6]

Last, we should recall the fact that the photon-number moments $\langle n^k(t)\rangle$ (including the Fano factor) contain no information about off-diagonal elements of the density operator. Therefore, the Fano-factor oscillations do not result from so-called "quantum coherence," but should be explained from a classical probability theory. In other words, the Fano factor evolves in time by referring only to the photon distribution $\rho_{nn}(t)$.

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