Stability and Mix in Spherical Geometry

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We consider a spherical system composed of N concentric fluid shells having perturbations of amplitude η_i at interface i, $i=1,2,\ldots,N-1$. For arbitrary implosion-explosion histories $R_i(t)$, we present the set of $N-1$ second-order differential equations describing the time evolution of the η_i which are coupled to the two adjacent $\eta_{i\pm1}$. We report analytical solutions for the $N=2$ and $N=3$ cases. We also present a model to describe the evolution of a turbulent mixing layer in spherical geometry when the interface between two fluids undergoes a constant acceleration or a shock.

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Several years ago Plesset published his analysis of the linear stability problem for a two-fluid spherical system.[1] In this Letter we extend his analysis to the N-fluid system much the way we have done earlier in plane geometry.[2] We also present a model for turbulent mix in spherical geometry. The issues of stability and mix in spherical geometry are important for inertial-confinement-fusion capsules[3] and for astrophysical applications such as supernova explosions.[4]

We believe that a recent treatment[5] of the N-fluid spherical problem is in error for two major reasons: First, the assumption of a constant acceleration g and the assumption of an exponentially growing perturbation η are incompatible in spherical geometry—one or the other can be chosen, but not both. Second, the assumption made in Ref. 5 that all the interfaces in the N-fluid system have a constant acceleration violates mass conservation.

In Fig. 1 we show the system and some of our notation. The last fluid of density ρ_N is assumed to have $R_N=\infty$. The radii R_i have a perturbation of small amplitude η_i so that the interface between two fluids of density ρ_i and ρ_{i+1} is at $r_i=R_i+\eta_i Y_{n,m}(\theta,\varphi)$. Given the (arbitrary) bulk radial histories $R_i(t)$, we find the evolution of the perturbation amplitude $\eta_i(t,n,m)$, $i=1,2,\ldots,N-1$.

Evolution equation.—The extension of Plesset's two-fluid analysis to the N-fluid system is tedious but straightforward, and we present only our final results (details will appear elsewhere[6]). As in the planar case, we find that the evolution of η_i depends on the two adjacent $\eta_{i\pm1}$. In this way all interfaces are coupled to each other.

We make the following physical assumptions: linearity ($n\eta_i \ll R_i$), incompressibility, no surface tension, no viscosity, and no heat transfer. Using continuity of the pressure at the perturbed interface $R_i+\eta_i Y_{n,m}$, we find the evolution equation for the amplitude η_i:

$$(\rho_{i+1}-\rho_i)\frac{d}{dt}(R_i^2\dot{R}_i\eta_i) = R_i^{n+2}\left[\rho_{i+1}\frac{dB_{i+1}}{dt}-\rho_i\frac{dB_i}{dt}\right]+(n\rightarrow -n-1)\,, \tag{1}$$

where

$$B_i=\frac{-(R_i)^{-n-1}}{n[1-(R_{i-1}/R_i)^{2n+1}]}\frac{d}{dt}(R_i^2\eta_i)+(i\leftrightarrow i-1)\,. \tag{2}$$

In Eq. (1) the notation $+(n\rightarrow -n-1)$ means adding to the first term on the right-hand side of that equation an identical term in which n is replaced by $-n-1$. This replacement must be done in B_i, B_{i+1}, and, of course, in the factor R_i^{n+2} appearing explicitly in that term. In Eq. (2) the notation $+(i\leftrightarrow i-1)$ means adding to the first term on the right-hand side of that equation an identical term in which i and $i-1$ are interchanged. For example, the ratio R_{i-1}/R_i in the denominator becomes R_i/R_{i-1} and the remaining R_i and η_i become R_{i-1} and η_{i-1}.

In deriving the above equations we have used mass conservation via $R_i^2\dot{R}_i=R_{i-1}^2\dot{R}_{i-1}$ because (shell mass) $\sim$(shell volume)$\sim R_i^3-R_{i-1}^3=$const. It is clear that if $\ddot{R}_i$ is constant then $\ddot{R}_{i-1}$ (or any other $\ddot{R}_j$) *cannot* be

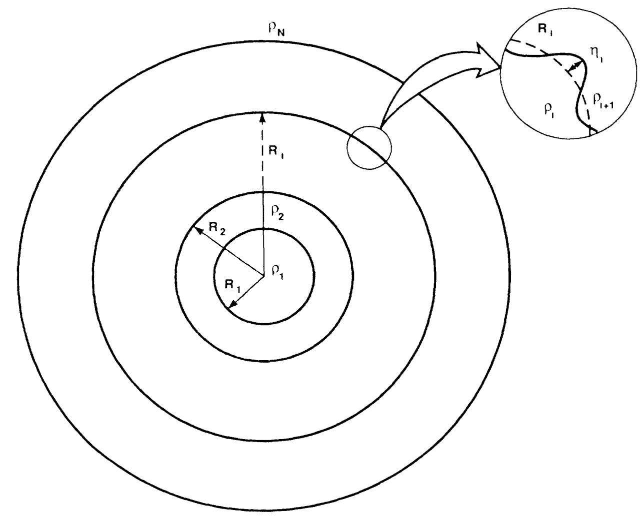

FIG. 1. N-fluid system in spherical geometry considered in this paper. The successive densities are $\rho_1,\rho_2,\ldots,\rho_{N-1},\rho_N$, and the radii are $R_1,R_2,\ldots,R_{N-1},\infty$. R_i is the average radius of the interface between the two fluids of densities ρ_i and ρ_{i+1}, and has a perturbation of amplitude η_i.

constant. By assuming a constant acceleration at each interface the authors of Ref. 5 violate mass conservation in each shell. This is a strictly spherical effect.

Special care must be exercised in the first and last regions which cover $0 \le r \le R_1$ and $R_{N-1} \le r \le \infty$, where the velocity potential is proportional to r^n and r^{-n-1}, respectively (it is a linear combination of the two in the intermediate shells). Denoting the $(n \to -n-1)$ terms in Eq. (1) by C_i, the conditions in the first and last regions read $C_1 = 0$ and $B_N = 0$, respectively.

Equation (1) exhibits an interesting symmetry for $N \ge 3$: The evolution equations are invariant under $n \leftrightarrow -n-1$ provided that $\rho_1 = \rho_N = 0$. This proviso is needed because we need to set $C_1 = B_N = 0$, and the resulting two equations in the first and last regions are symmetric under $n \leftrightarrow -n-1$ if and only if $\rho_1 = \rho_N = 0$. We conclude that in a system with an arbitrary number of shells bounded by a vacuum on the inside and on the outside the evolution of perturbations is symmetric under $n \leftrightarrow -n-1$. In plane geometry this translates to symmetry under $k \leftrightarrow -k$, which we had not noticed earlier.[2]

N=2.—Equation (1) reduces to Plesset's equation which we write as

$$\frac{1}{R^3}\frac{d}{dt}\left(R^3\frac{d\eta}{dt}\right) - nA(n)\frac{\ddot{R}}{R}\eta = 0 \tag{3}$$

in which

$$nA(n) = \frac{(n^2+n+1)A - 2n - 1}{n + (1-A)/2}, \tag{4}$$

where A is the Atwood number, $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$. Note that $A(n) \to A$ as $n \to \infty$. This notation was chosen with an eye towards an easy transition to the planar limit ($n \to \infty$, $R \to \infty$, $n/R \to k$ finite).

Let us point out that for a spherical cavity $\rho_1 = 0$; hence, $A = 1$ and $nA(n)$ reduces to $n-1$ in which case Eq. (3) gives the well-known result for the stability of a spherical cavity[7] (the results of Ref. 5 do not reproduce this case).

We have found several analytic solutions to Eq. (3) which we classify as class A or class B. Class-A solutions are valid for arbitrary $R(t)$ histories but only specific choices of $nA(n)$. We have found solutions for $nA(n) = -2$ and $nA(n) = 0$. We label the modes satisfying $nA(n) = 0$ as n_{critical},

$$n_{\text{critical}} = [2 - A + (4 - 3A^2)^{1/2}]/2A, \tag{5}$$

because if $\dot{\eta}_0 = 0$ then $\eta(t, n_{\text{critical}}, m) = \eta_0$ for all $t > 0$; i.e., critical modes do not evolve with time for any implosion-explosion history $R(t)$. This follows from Eq. (3) which implies that if $nA(n)$ vanishes then $R^3\dot{\eta}$ is conserved. For example, $n_{\text{critical}} = 1$ if $A = 1$ and $n_{\text{critical}} = 10$ if $A = \frac{7}{37}$. Note that $n = 1$ corresponds only to a shift of origin for the implosion or explosion.

In contrast, class-B solutions are valid for arbitrary $nA(n)$ but specific $R(t)$ histories. We have found analytic solutions for four cases: (i) $R = R_0 e^{t/T}$, (ii) $R = R_0 \times (1 + t/T)^{1/3}$, (iii) $R = R_0 + \frac{1}{2}gt^2$, and (iv) $\ddot{R} = \Delta v \delta(t)$. Here we discuss only the last two cases which we refer to as a constant acceleration (Rayleigh-Taylor instability[8]) and a shock (Richtmyer-Meshkov instability[9]).

In plane geometry a constant g leads to perturbations growing exponentially in time. The same is not true here. When $R = R_0 + \frac{1}{2}gt^2$, Eq. (3) reduces to the hypergeometric equation

$$x(1-x)\frac{d^2\eta}{dx^2} + 6(\tfrac{1}{2} - x)\frac{d\eta}{dx} + 2nA(n)\eta = 0, \tag{6}$$

where $x = (t+\alpha)/2\alpha$, $\alpha^2 = -2R_0/g$. The solution to this equation is $\eta = F(a_+, a_-; 3; x)$ in the notation of Ref. 10, where

$$a_\pm = \tfrac{1}{2}[5 \pm \sqrt{25 + 8nA(n)}]. \tag{7}$$

As we mentioned earlier, a constant acceleration does *not* lead to an exponentially growing perturbation in spherical geometry. In fact, to get an exponentially growing perturbation ($\eta \sim e^{\gamma t}$) the radial history itself must be exponential [$R = R_0 e^{t/T}$, case (i); see Ref. 6].

In plane geometry a shock followed by a constant velocity leads to perturbations growing linearly in time. The same is not true here. Assuming that $\dot{\eta}_0 = \dot{R}_0 = 0$ (these are preshock values) and that postshock $\dot{R} = R_0/T$, we find

$$\eta(t) = \eta_0\{1 + \tfrac{1}{2}nA(n)[1 - (R_0/R)^2]\} \tag{8}$$

(the general case is given in Ref. 6). Since $1 - (R_0/R)^2 = t(t+2T)(t+T)^{-2}$, $\eta(t)$ grows linearly in time only during $t \ll |T|$. In deriving Eq. (8) we have adopted Richtmyer's technique of treating a shock as an instantaneous acceleration of incompressible fluids.

In general, we expect an incompressible treatment such as ours to be valid as long as the sound speed c_s of the fluids is much larger than the velocity of the perturbations, i.e., $c_s^2 \gg R\ddot{R}/n$ (in plane geometry $c_s^2 \gg g/k$). Shocks require a careful treatment. Richtmyer[9] described how the amplitudes and the densities must be modified to obtain agreement with his numerical calculations on fully compressible fluids in plane geometry. Until new numerical calculations are carried out in spherical geometry we suggest following Richtmyer's prescription, i.e., using postshock amplitudes and densities.

Equation (8) predicts that the perturbation will change phase if $nA(n)$ is positive and the radius is imploding, or if $nA(n)$ is negative and the radius is exploding. For example, if $nA(n) = 50$ then η will go through zero when $R/R_0 = \sqrt{25/26} \approx 98\%$. Clearly, the time when a perturbation goes through zero is independent of its initial amplitude η_0.

It is possible to freeze an amplitude, i.e., set $\dot{\eta}(\text{postshock}) = 0$, if a second shock arrives at the right time or, equivalently, at the right radius R_s. Denoting the preshock and postshock radial velocities by v_i and v_f, respectively, freeze-out occurs if

$$\left(\frac{R_s}{R_0}\right)^2 = \frac{\frac{1}{2}nA(n)(v_f/v_i - 1) - 1}{(v_f/v_i - 1)[1 + \frac{1}{2}nA(n)]}. \tag{9}$$

Note that there is again no dependence on the initial amplitude η_0. The same phenomenon occurs in plane geometry[2] if we assume, as we have done here, that the initial conditions are set by a first shock (see Refs. 2 and 6 for general conditions on freeze-out).

N=3.—The evolution equations given in Eq. (1) form a system of $N-1$ coupled linear differential equations which in general must be solved numerically. Even for $N=3$ the two resulting equations are quite complicated, and we could find no analytic solutions except for the case $\rho_1=\rho_3=0$ and in the limit of a thin shell $\Delta R/R \ll 1$, where $\Delta R = R_2 - R_1$ and $R=(R_2+R_1)/2$. Defining $\Delta\eta = \eta_2-\eta_1$ and $\eta=(\eta_2+\eta_1)/2$, we find

$$\frac{d^2\eta}{dt^2} = -\frac{\ddot{R}}{\Delta R}\Delta\eta\,, \tag{10a}$$

$$\frac{d^2}{dt^2}\left(\frac{R\Delta\eta}{\Delta R}\right) = \ddot{R}\left(3\frac{\Delta\eta}{\Delta R} - (n+2)(n-1)\frac{\eta}{R}\right). \tag{10b}$$

A fair amount of algebra was needed to obtain the above two equations from Eq. (1). As expected they are invariant under $n \leftrightarrow -n-1$. These coupled second-order differential equations can be solved analytically if the motion starts with a shock followed by a constant radial velocity, i.e., $\ddot{R} = \Delta v \delta(t) = (R_0/T)\delta(t)$. We find

$$\eta(t) = \eta_0 - \Delta\eta_0 (R_0/\Delta R_0) t/T\,, \tag{11a}$$

$$\Delta\eta(t) = \left[\Delta\eta_0\left(1+\frac{3t}{T}\right) - (n+2)(n-1)\eta_0\frac{\Delta R_0}{R_0}\frac{t}{T}\right] \times \left(1+\frac{t}{T}\right)^{-3}. \tag{11b}$$

If the initial perturbations $\eta_1(0)$ and $\eta_2(0)$ are equal, then their sum does not evolve with time [Eq. (11a) with $\Delta\eta_0=0$]. Their difference $\Delta\eta$, however, evolves with time no matter what the initial conditions read. A second shock may arrive later and, if properly timed, can freeze out η_1 or η_2, but *not* both.

Numerical examples for $N=2$ and $N=3$ are given in Ref. 6. We find that convergence enhances the growth of the perturbations, while divergence curtails it: In an implosion (explosion) perturbations grow faster (slower) than the exponential growth in plane geometry. In an implosion typical of inertial-confinement-fusion capsules with a convergence ratio $R_0/R_{\rm final}=25$ and a density ratio $\rho_2/\rho_1=10$, we found $n=50$ perturbations $[nA(n) \approx 40]$ growing extremely large during the final decelerating stage of the implosion.

Another important difference between planar and spherical geometries is the following: In planar geometry the effectiveness of feedthrough or interface coupling remains constant because the shell thickness remains constant; in spherical geometry, on the other hand, the shell thickens up (thins out) during an implosion (explosion); hence feedthrough becomes less (more) effective during the later stages. This is clearly seen in our numerical examples with $N=3$. We should add, however, that the beneficial effect of a shell that thickens up during implosion is more than offset by the rapid growth of the perturbation unless, of course, nonlinear effects intervene to slow down the growth.

Turbulent mix.—Naturally occurring surface finishes involve multimode perturbations with more or less random amplitudes and phases. As they grow large and enter the nonlinear regime such random perturbations can be described as forming a mixing layer of width h. Here we propose a model to describe the time evolution of h at the interface of two fluids ($N=2$). The emphasis is on spherical geometry, as experiments and models of turbulent mix in plane geometry already exist.[11,12]

The model we propose is based on Eq. (3). We assume that h obeys a similar equation after taking the limit $n\rightarrow\infty$, $\eta\rightarrow 0$, with $n\eta/R = c =$(a dimensionless constant), i.e.,

$$\frac{1}{R^3}\frac{d}{dt}\left(R^3\frac{dh}{dt}\right) - cA\ddot{R} = 0\,. \tag{12}$$

The physical significance of the model is that a turbulent mix in spherical geometry is driven by perturbations of wavelength much shorter than the radius and having correspondingly smaller amplitudes in such a way that the ratio between wavelength and amplitude remains constant. Strictly speaking, we only require that $A(n)\rightarrow A$ and $n\eta/R \rightarrow c$ with no additional constraints on n or η. In plane geometry the model reduces to $k\eta = 2\pi\eta/\lambda = c$, again with no constraints on k or η separately.

The solution $h(t)$ to Eq. (12) can be written down explicitly for arbitrary radial histories $R(t)$ and initial values h_0 and dh_0/dt. For simplicity we assume here that $h_0 = dh_0/dt = 0$ and write down $h(t)$ for the case of a constant acceleration

$$h = \frac{cAgt^2}{70}\left[5 + 16\frac{R_0}{R} + 8\left(\frac{R_0}{R}\right)^2 + \frac{6}{R/R_0 - 1}\ln\left(\frac{R}{R_0}\right)\right], \tag{13}$$

and for the case of a shock followed by a constant velocity,

$$h = \tfrac{1}{2} cA\Delta v t[R_0/R + (R_0/R)^2]\,. \tag{14}$$

If we take the planar limits of the above three equations, we get

$$\frac{d^2h}{dt^2} - cAg = 0\,, \tag{15}$$

$$h = \tfrac{1}{2} cAgt^2\,, \tag{16}$$

$$h = cA\Delta v t\,. \tag{17}$$

Note that the model [Eq. (12)] proposed here is very tightly constrained because it predicts h for constant accelerations and shocks in spherical and planar geometries and involves only one constant c. Read and Youngs[11] have carried out experiments with constantly accelerating fluids in plane geometry and their results agree with Eq. (16). Since they find $h \approx 0.07Agt^2$, where h is the mixing width into the heavier fluid,[11] we can "fix" our constant by taking $c \approx 0.14$. Equation (17) predicts $h \approx 0.14A\Delta vt$ for a shock, a form that we suggested earlier.[12] It remains to be verified experimentally.

The above equations predict a faster growth rate at an interface with a large density gradient or a large density discontinuity because as $\rho_2/\rho_1 \to \infty$, $A \to 1$. At an accelerating interface, the mix is expected to grow only if g is directed from the lower to the higher density fluid, i.e., $A > 0$ in our notation. At a shocked interface, the mix is likely to grow for shocks in either direction ($A > 0$ or $A < 0$), an expectation based on our analysis of the linear regime [Eq. (18)] as well as extrapolations from planar experiments.

Equations (12)-(17) suggest that the turbulence is largely independent of initial conditions and therefore the evolution of the mixing width h is insensitive to the size of the assumed perturbations. An initial value of h_0 can be accommodated in integrating Eq. (12) (see Ref. 6). However, memory of initial conditions is lost as the mixing width grows in time. There is experimental evidence[11] for this behavior in plane accelerating interfaces. No direct experimental data are available for shocked interfaces or in spherical geometry.

From Eqs. (13)-(17) we see that in our model h_{spher} factorizes into h_{planar} and a geometrical factor (GF) which is a function of the dimensionless variable R/R_0 only, i.e., we can write $h_{\text{spher}} = h_{\text{planar}} \times \text{GF}$. We find that GF is a decreasing function of R/R_0 (see Fig. 2) implying that the mixing width evolves faster during an implosion (slower during an explosion) than in plane geometry. During the late states of an explosion, i.e., as $R/R_0 \to \infty$, the GF for a constant acceleration becomes asymptotic to $\frac{1}{7}$, and hence $h_{\text{spher}} \to \frac{1}{14} cAgt^2 \approx 0.01 \times Agt^2$, while for a shock the GF decreases in such a way that the mixing width becomes asymptotic to a constant value, $h_{\text{spher}} \to \frac{1}{2} cAR_0 \approx 0.07AR_0$.

We emphasize that unlike the linear analysis which is based on first principles, Eq. (12) is only a model.

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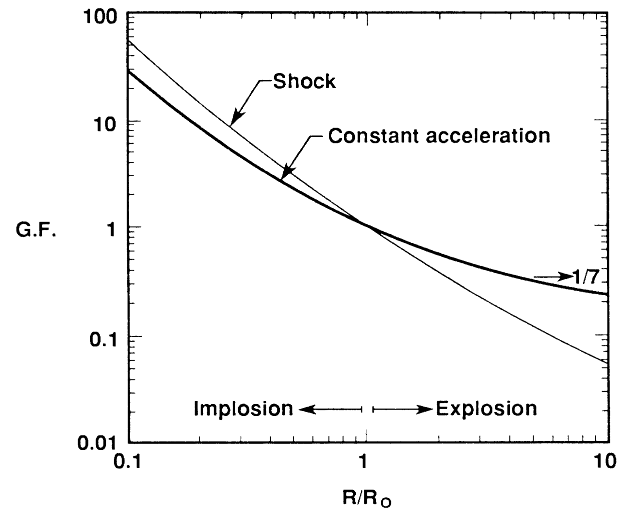

FIG. 2. Geometrical factors for a constant acceleration and a shock. The GF's relate spherical and planar mixing widths via $h_{\text{spher}} = h_{\text{planar}} \times \text{GF}$.

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