Quantum Algebra as the Dynamical Symmetry of the Deformed Jaynes-Cummings Model

M. Chaichian,[(1),(a)] D. Ellinas,[(2)] and P. Kulish[(3),(b)]

[(1)]*Department of High Energy Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland*
[(2)]*Department of Theoretical Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland*
[(3)]*Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland*
(Received 13 April 1990)

The q-deformations of the quantum harmonic oscillator are used to describe the generalized Jaynes-Cummings model (JCM) by using the q-analog of the Holstein-Primakoff realization of su(1,1). The corresponding dynamical symmetry is described by a quantum algebra. The q-analogs of the Barut-Girardello and the Perelomov coherent states are introduced and the expectation value of σ_3 is calculated. The periodic revivals of the generalized JCM are destroyed more for increasing deformation parameter q. The deformed original JCM in the rotating-wave approximation can be described by $u(1|1)_q$, while its relaxation extends the dynamical algebra to the $osp(2|2)_q$ quantum superalgebra.

PACS numbers: 42.50.Md, 03.65.Fd, 11.30.Na

The Jaynes-Cummings model[1] (JCM) idealizes the interaction of matter with electromagnetic radiation by a simple Hamiltonian of a two-level atom coupled to a single bosonic mode. Despite its simplicity, the dynamics predicted by the model has been supported in recent Rydberg maser experiments.[2] On the theoretical side, the JC Hamiltonian has been generalized in quantum optics in different directions. Some of them which concern us here are the one with intensity-dependent coupling,[3] the Hamiltonian which describes a two-photon transition,[4] and the one which relates the JC Hamiltonian with the $u(1|1)$ and the $osp(2|2)$ superalgebras.[5]

On the other hand, the recent development of quantum groups[6] has motivated great interest in q-deformed algebraic structures, in particular the q-oscillators (cf. Ref. 7). In this framework there have already been examples to treat solvable models which permit the application of quantum algebras.[8]

Here we undertake an analogous task in the framework of the optical JC Hamiltonian. Initially, there is the fundamental problem of how one chooses to deform the JC Hamiltonian. We take the JC Hamiltonian with an intensity-dependent coupling constant.[3] The field operators of this Hamiltonian are identified as the elements of the su(1,1) algebra in the Holstein-Primakoff (HP) realization. By constructing the q-deformed analog of the HP realization, we generalize the JC Hamiltonian and connect it with the quantum $su(1,1)_q$ algebra. In physical terms this generalization permits us to introduce an additional parameter q into the JC Hamiltonian with an intensity-dependent coupling. The nature of this parameter can only be speculated upon at the present. However, its effect and range of influence can be traced from the changes of JCM dynamics exemplified by the time evolution of the population inversion $\langle\sigma_3(t)\rangle \equiv \langle\Psi(t)|\sigma_3|\Psi(t)\rangle$.

In the interaction picture and on resonance, the JC Hamiltonian in rotating-wave approximation (RWA) is given by ($\hbar=1$)

$$H_{\rm int}=\lambda(b^+\sigma^- + b^-\sigma^+) \tag{1}$$

and the free part

$$H_0=\omega(b^+b^- + \tfrac{1}{2}) + \tfrac{1}{2}\omega_0\sigma_3 \quad (\omega=\omega_0=1)\,, \tag{2}$$

where σ_i are the Pauli matrices. The symmetry of the total Hamiltonian is described by the Heisenberg and su(2) algebras. Introducing an intensity-dependent coupling,[3] one obtains

$$H'_{\rm int}=\lambda(\sqrt{N}b^+\sigma^- + b^-\sqrt{N}\sigma^+)\,, \tag{3}$$

where N is the number operator for the bosonic mode.

Such an intensity-dependent coupling allows for interactions of the two-level atom with the field, the strength of which depends on the number of photons present in the cavity during the flight time of the atom through this cavity [cf. also the comment in the third paragraph after Eq. (18)].

The dynamical algebra now becomes $su(1,1)\oplus su(2)$ if one identifies in the HP realization[3(c)] the generators of the su(1,1) algebra by $L_+=\sqrt{N}b^+$, $L_-=b^-\sqrt{N}$, $L_0=N+\frac{1}{2}$.

The deformation of the JC Hamiltonian (3) yields

$$H^{(q)}_{\rm int}=\lambda(\sqrt{[N]}a^+\sigma^- + a^-\sqrt{[N]}\sigma^+)\,, \tag{4}$$

where we use the notation $[x]=(q^x-q^{-x})/(q-q^{-1})$, q is the deformation parameter, and $a^\pm$ are the q-deformed bosonic oscillators which satisfy the relations[7]

$$a^-a^+ - qa^+a^- = q^{-N}, \quad [N,a^\pm]=\pm a^\pm\,. \tag{5}$$

We now write the q-analog of the HP realization of the quantum $su(1,1)_q$ algebra as

$$K_+=\sqrt{[N]}a^+, \quad K_-=a^-\sqrt{[N]}, \quad K_0=N+\tfrac{1}{2}\,, \tag{6}$$

with commutation relations (CR)

$$[K_-,K_+]=[2K_0],\quad [K_0,K_\pm]=\pm K_\pm . \tag{7}$$

We rewrite the quantum version of the JC Hamiltonian in the form

$$H_{\rm int}^{(q)}=\lambda(K_+\sigma^- + K_-\sigma^+)\,, \tag{8}$$

which manifests now that the dynamical algebra is $su(1,1)_q \oplus su(2)$. By expressing the quantum $su(1,1)_q$ generators, $K_\pm$, in this representation in terms of their nondeformed counterparts, $L_\pm$, i.e.,

$$K_+=L_+\frac{[N+1]}{N+1},\quad K_-=\frac{[N+1]}{N+1}L_-,\quad K_0=L_0\,, \tag{9}$$

we can give the Hamiltonian in the equivalent form

$$H_{\rm int}^{(q)}=\lambda\left\{L_+\sigma^-\frac{[N+1]}{N+1}+\frac{[N+1]}{N+1}L_-\sigma^+\right\}, \tag{10}$$

which reveals the meaning of q-deformation as the intensity-dependent coupling with an additional parameter q.

Since the Heisenberg equations of motion need not be deformed,[9] one has the evolution operator of this system in the usual form, $U(t)=\exp(-itH_{\rm int}^{(q)})$, which can be written as[10]

$$U(t)=\begin{pmatrix} \cos(\lambda t\sqrt{K_-K_+}) & -i\dfrac{\sin(\lambda t\sqrt{K_-K_+})}{\sqrt{K_-K_+}}K_- \\ -iK_+\dfrac{\sin(\lambda t\sqrt{K_-K_+})}{\sqrt{K_-K_+}} & \cos(\lambda t\sqrt{K_+K_-}) \end{pmatrix}, \tag{11}$$

with $K_+K_-=[N]^2$ and $K_-K_+=[N+1]^2$.

For an initial state $|\Psi(0)\rangle=|+\rangle\otimes|\psi\rangle$, where $|+\rangle$ denotes the fermionic ground state and $|\psi\rangle\in\mathcal{H}_F$, the time evolution of the population inversion can be calculated to give

$$\langle\sigma_3(t)\rangle=\sum_{n=0}^{\infty}\cos(2\lambda t[n+1])|\langle\psi|n\rangle|^2 . \tag{12}$$

In deforming the JC Hamiltonian we used q-oscillators without changing the free Hamiltonian, although in H_0 one could, instead of $b^+b^-=N$, use the term $a^+a^-=[N]$ with q-bosons. In this case it would be possible to obtain analogous expressions for $\langle\sigma_3(t)\rangle$ through more cumbersome algebraic calculations.

To probe the dynamics of the deformed JCM, we set the initial state of the q-bosonic mode as the deformed analogs of the Glauber (G), Barut-Girardello (BG), and Perelomov (P) coherent states (CS). The occurrence of the q-coherent states rather than the normal ones appears naturally since the theory is q-deformed.

Specifically, the analog of the GCS is defined as[7(b)]

$$\begin{aligned}|\alpha\rangle_q&=N_G^{-1/2}\exp_q(\alpha a^+)|0\rangle\\ &=N_G^{-1/2}\sum_{n=0}^{\infty}\alpha^n\frac{(a^+)^n}{[n]!}|0\rangle\\ &=N_G^{-1/2}\sum\frac{\alpha^n}{\sqrt{[n]!}}|n\rangle\,,\end{aligned} \tag{13}$$

where the normalization factor is $N_G=\exp_q|\alpha|^2$ and the notion of q-exponential[11] has been used. The same states can be generated using the usual exponential and a new operator T:[12]

$$|\alpha\rangle_q=N_G^{-1/2}\exp(\alpha T)|0\rangle=N_G^{-1/2}D(\alpha)|0\rangle\,,$$
$$T=a^+\frac{N+1}{[N+1]}=b^+\left(\frac{N+1}{[N+1]}\right)^{1/2}, \tag{14}$$

where the displacement operator $D(\alpha)$ is now a group element.

For the BGCS we define the q-analog as

$$K_-|z\rangle_q=z|z\rangle_q\,,$$
$$|z\rangle_q=N_{BG}^{-1/2}\sum_{n=0}^{\infty}z^n\left(\frac{\Gamma_q(2k)}{[n]!\Gamma_q(2k+n)}\right)^{1/2}|k;n\rangle\,, \tag{15}$$
$$K_0|k;n\rangle=(k+n)|k;n\rangle\,,$$

where we use the q-gamma function[11] and the normalization factor $N_{BG}={}_0F_1(k;|z|^2)_q$.

By analogy with (13) and taking into account (6), we define the q-analog of PCS by

$$\begin{aligned}|\xi\rangle_q&=N_P^{-1/2}\exp_q(\xi K_+)|0\rangle\\ &=N_P^{-1/2}\sum_{n=0}^{\infty}\xi^n\left(\frac{\Gamma_q(2k+n)}{[n]!\Gamma_q(2k)}\right)^{1/2}|k;n\rangle\,,\end{aligned} \tag{16}$$
$$N_P=(1-|\xi|^2)^k .$$

Seeking to define this state in terms of usual exponential, we set

$$|\xi\rangle_q=N_P^{-1/2}\exp(\xi T')|0\rangle\,, \tag{17}$$

which yields for T' the form

$$T'=K_+(N+1)/[N+1]=L_+ .$$

Hence T' is identical to the usual raising operator of $su(1,1)$. Thus the PCS are not deformed.

Considering the three sets of CS introduced above as the initial states for the bosonic mode, we evaluate the population inversion as

$$\langle\sigma_3(t)\rangle_j=N_j^{-1}\sum_{n=0}^{\infty}\cos(2\lambda t[n+1])W_j(n)\,, \tag{18}$$

with j=G, BG, and P and $W_G=|\alpha|^{2n}/[n]!$, $W_{BG}=|z|^{2n}/([n]!)^2$, and $W_P=|\xi|^{2n}$ for $k=\frac{1}{2}$.

To examine the influence of deformation on the dynamics, we concentrate here on the first case with the q-Glauber coherent-state preparation of the bosonic mode. Formally, the series (18) giving $\langle\sigma_3(t)\rangle_G$ is similar to the nondeformed ($q=1$) case which can be summed up[3] to give a closed-form solution and exhibits periodic collapse and revivals. However, in the deformed case the presence of q-quantities excludes a closed-form expression.

In Fig. 1 we display the numerical calculation of the population inversion for the values of $q=1$ (solid line), 1.01 (dotted line), and 1.03 (dashed line). One can see that in addition to the disappearance of periodicity in the deformed case ($q>1$), the more deformed the model, the longer the departure from perfect revivals is, and, besides, the more spread the revivals acquire as time increases. This situation is akin to the original coherent[13] JCM of Eq. (1), i.e., without an intensity-dependent coupling.

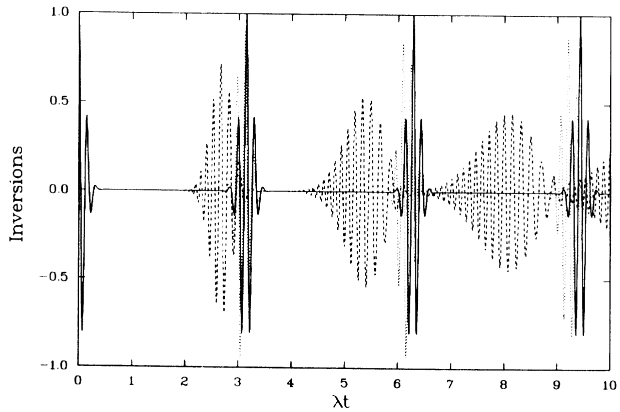

FIG. 1. The population inversions $\langle\sigma_3(t)\rangle$ of the generalized Jaynes-Cummings model as a function of time for different values of deformation parameter q [see the text after Eq. (18)].

Let us also remark on the case of the two-photon transition in the JCM, $H_{\rm int}=\lambda(b^{+2}\sigma^-+\sigma^+b^{-2})$, which utilizes the $L_\pm=\frac{1}{2}b^{\pm 2}$ realization of su(1,1) algebra. This gives an exact solution

$$\langle\sigma_3(t)\rangle_G=e^{-|\alpha|^2}\sum_{n=0}^{\infty}\cos(2\lambda t\sqrt{(n+1)(n+2)})\frac{|\alpha|^{2n}}{n!},$$

which in the large-number-of-photons limit $\bar{n}=|\alpha|^2\gg 1$ becomes

$$\langle\sigma_3(t)\rangle_G\simeq e^{-|\alpha|^2}\sum_{n=0}^{\infty}\cos(2\lambda t(n+1))\frac{|\alpha|^{2n}}{n!},$$

which in turn is the solution of the intensity-dependent-coupling JCM. Similarly, the q-deformed version of the two-photon JCM, $H_{\rm int}^{(q)}=\lambda(a^{+2}\sigma^-+\sigma^+a^{-2})$, which utilizes the $K_\pm=(q+q^{-1})^{-1}a^{\pm 2}$ realization of the su$(1,1)_q$ quantum algebra, permits in the large-$\bar{n}$ limit the approximation

$$\langle\sigma_3(t)\rangle_G=(\exp_q|\alpha|^2)^{-1}\sum_{n=0}^{\infty}\cos(2\lambda t\sqrt{[n+1][n+2]})\frac{|\alpha|^{2n}}{[n]!}\simeq(\exp_q|\alpha|^2)^{-1}\sum_{n=0}^{\infty}\cos(2q^{1/2}\lambda t[n+1])\frac{|\alpha|^{2n}}{[n]!},$$

which is the expression for the deformed intensity-dependent-coupling JCM we examined above in Eq. (18) with scaled λ.

Deformation of JCM can also be performed following the u(1|1)-superalgebra formulation of the model.[5] We show that the deformation invokes the quantum superalgebra $u(1|1)_q$ which now describes the dynamical symmetry of the model. Indeed, the deformed JC Hamiltonian

$$H^{(q)}=\omega(P_++P_-)+\omega_0(P_+-P_-)+\lambda(\gamma V_-+V_+\bar{\gamma}),\quad(19)$$

with γ and $\bar{\gamma}$ as Grassmann variables, is written as an element of the $u(1|1)_q$ quantum algebra in the realization $V_\pm=a^\mp f^\pm$, $P_+=\frac{1}{2}(N+M)$, $P_-=\frac{1}{2}(N-M+1)$. The q-fermions[7(c)] $f^\pm$ have been used, for generality, with CR

$$f^-f^++qf^+f^-=q^M,\quad [M,f^\pm]=\pm f^\pm,\quad(20)$$

while $f^+f^-=[M]=M$ and $f^-f^+=[1-M]=1-M$. In particular, in JCM the (simplest) realization $f^\pm=\sigma^\pm$ occurs. The $u(1|1)_q$ dynamical quantum superalgebra obeys the CR

$$[V_+,V_-]_+=[2P_+],\quad [V_\pm,P_-]=\pm V_\pm,$$
$$[V_\pm,P_+]=0.\quad(21)$$

Relaxing the RWA, we add to the JC Hamiltonian (1) the energy-nonconserving term $H'^{(q)}=\theta\tilde{V}_-+\tilde{V}_+\bar{\theta}$, where $\theta,\bar{\theta}$ are additional Grassmann variables and $\tilde{V}_\pm=a^\mp f^\mp$ are odd generators which together with $P_\pm$, $V_\pm$, and $K_\pm=(q+q^{-1})a^{\pm 2}$ form the $osp(2|2)_q$ quantum superalgebra [in the sense of deformed universal enveloping superalgebra $U_q osp(2|2)$; see also Ref. 14]. This is the new dynamical algebra of the system with CR

$$[\tilde{V}_+,\tilde{V}_-]_+=[2P_-],\quad [\tilde{V}_\pm,P_+]=\pm\tilde{V}_\pm,$$
$$[\tilde{V}_\pm,P_-]=0.\quad(22)$$

In the nondeformed case the spectrum of the Hamiltonian is obtained by the adjoint action of the Osp(2|2) group on the JC Hamiltonian. In the deformed case there exists no adjoint action of the corresponding quantum group due to the peculiarity of this object.[15] Therefore, we can diagonalize the Hamiltonian only by employing the eigenvalue equations involving Grassmann variables.

As demonstrated specifically on the generalized JCM, q-deformation can seemingly be applied to any physical problem. This deformation introduces, in addition to the parameters already present in the original problem, a new q parameter to which it is of fundamental importance to give a physical interpretation. Such a question has been addressed in the work on, e.g., q-oscillators.[7] It is also known that the fundamental constants $\hbar$ and c can be considered as deformations of classical mechanics and of Galilean invariance. Hence the parameter q opens a new possibility to deform a modern physical theory.[16] If, on the same footing as the other fundamental constants of nature $\hbar$ and c, one would like to have another dimensional constant in order to make the deformation parameter q dimensionless, one is forced to introduce a new fundamental constant of dimension length (or equivalently mass).

Since q-deformed algebras have originally emerged out of unrealistic integrable systems in one-dimensional space or in a chain, we have tried here to q-deform a simple model which at the same time would permit an experimental verification in one way or another. Besides, it is noticeable that the q-deformed version of one of the fundamental theoretical paradigms of quantum optics, the Jaynes-Cummings model and its generalization, allows one to work out all the details of the model as is the case in the original nondeformed version.

It is our pleasure to thank L. D. Faddeev and S. Stenholm for useful discussions.

---

[a]Address after 1 September 1990: Theory Division, CERN, CH-1211 Geneva 23, Switzerland.

[b]On leave of absence from Steklov Mathematical Institute of the U.S.S.R. Academy of Sciences, Leningrad Branch, 191011 Leningrad, U.S.S.R.

[1]E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

[2]D. Meschede, H. Walther, and G. Muller, Phys. Rev. Lett. **54**, 551 (1985); G. Rempe, H. Walther, and N. Klein, Phys. Rev. Lett. **58**, 353 (1987).

[3](a) B. Buck and C. V. Sukumar, Phys. Lett. **81A**, 132 (1981); (b) J. Phys. A **17**, 885 (1984); (c) V. Bužek, Phys. Rev. A **39**, 3196 (1989).

[4]B. Buck and C. V. Sukumar, Phys. Lett. **83A**, 211 (1981); S. Singh, Phys. Rev. A **25**, 3206 (1982); C. C. Gerry, Phys. Rev. A **37**, 2683 (1988).

[5]C. Buzano, M. G. Rasetti, and M. L. Rastello, Phys. Rev. Lett. **62**, 137 (1989).

[6]See, e.g., V. G. Drinfeld, in Proceedings of the International Congress of Mathematicians, Berkeley, 1986 (unpublished), Vol. 1, p. 798; M. Jimbo, Lett. Math. Phys. **11**, 247 (1986).

[7](a) A. J. Macfarlane, J. Phys. A **22**, 4581 (1989); (b) L. C. Biedenharn, J. Phys. A **22**, L873 (1989); (c) M. Chaichian and P. Kulish, Phys. Lett. B **234**, 72 (1990).

[8]V. Pasquier and H. Saleur, Nucl. Phys. **B330**, 523 (1990).

[9]M. Chaichian and D. Ellinas, J. Phys. A **23**, L291 (1990).

[10]S. Stenholm, Phys. Rep. **43**, 151 (1973).

[11]H. Exton, *q-Hypergeometric Functions and Applications* (Horwood, Chichester, 1983).

[12]P. P. Kulish and E. V. Damaskinsky, J. Phys. A **23**, L415 (1990).

[13]J. H. Eberly, N. B. Narozhny, and J. J. Sanchez-Mondragon, Phys. Rev. Lett. **44**, 1323 (1980); N. B. Narozhny, J. J. Sanchez-Mondragon, and J. H. Eberly, Phys. Rev. A **23**, 236 (1981).

[14]M. Chaichian, P. Kulish, and J. Lukierski, Phys. Lett. B **237**, 401 (1990).

[15]L. A. Takhtajan, Adv. Stud. Pure Math. **19**, 435 (1989).

[16]L. D. Faddeev, in *Integrable Systems,* edited by S. Xing-Chang, Nankai Lectures on Mathematical Physics (World Scientific, Singapore, 1990).