# More Sum Rules for Quark and Lepton Masses

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Sum rules for quark and lepton masses are derived from the Ward identity of Chanowitz and Ellis for the vertex function of the trace of the energy-momentum tensor and two axial-vector currents, and the partially conserved axial-vector current hypothesis. The sum rules indicate, among other things, that the constituent masses of the $u$ and $d$ quarks and those of the techniquarks, if any, are about 300 MeV and 600 GeV, respectively.

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In the unified gauge theory of Glashow, Salam, and Weinberg (GSW) for electroweak interactions,[1] all the masses of the weak bosons ($W^{\pm}$ and $Z$), the physical Higgs scalar ($H$), and the quarks and leptons ($f$'s) are generated by the single vacuum expectation value ($v$) of the Higgs scalar [$\langle\phi^0\rangle_0 = v/\sqrt{2}$, where $v=(\sqrt{2}G_F)^{-1/2} \cong 246$ GeV] as

$$m_W = m_Z\cos\theta_W = gv/2, \qquad m_H=\sqrt{2\lambda}v, \quad m_f = G_f v/\sqrt{2}, \tag{1}$$

where $\theta_W$ is the weak mixing angle and $g$, $\lambda$, and $G_f$'s are the weak gauge coupling constant, the Higgs-boson quartic coupling constant, and the Higgs-boson Yukawa coupling constants of quarks and leptons. Since these coupling constants are arbitrary, the masses of $W^{\pm}$, $Z$, $H$, and $f$s are not related to each other in the theory. This is one of the reasons why the theory must be taken not as the final theory but probably as a low-energy effective theory which can be reproduced with much less arbitrary parameters by a more fundamental theory. Such a more fundamental theory must clarify how these couplings constants, $g$, $\lambda$, and $G$'s, or these masses, $m_W$, $m_Z$, $m_H$, and $m_f$'s, are related to each other. In order to find how symmetry is broken spontaneously and how particle masses appear to be related to each other, it is always more than just instructive to investigate a model of the Nambu-Jona-Lasinio (NJL) type.[2]

In fact, in 1977 we found[3] that the unified model of the NJL type not only reproduces the GSW theory as an effective theory at low energies but also leads to the following sum rules for quark and lepton masses:

$$\left(\frac{\sum_f m_f^2}{8N_g}\right)^{1/2} = \frac{m_W}{\sqrt{3}}, \tag{2}$$

$$\left(\frac{\sum_f m_f^4}{\sum_f m_f^2}\right) = \frac{m_H}{2}, \tag{3}$$

where $m_W$ and $m_H$ are the masses of the $W^{\pm}$ bosons and Higgs scalar, $N_g$ is the total number of quark and lepton generations, and the summations are taken over all $8N_g$ quarks and leptons. If $N_g=3$, we can predict from the first sum rule (2) the mass of the top quark as[4]

$$m_t \cong (\tfrac{8}{3})^{1/2} m_W \cong 131 \text{ GeV} \quad \text{for } m_W = 80 \text{ GeV}, \tag{4}$$

and further, from the second sum rule (3), the mass of the Higgs scalar as[4]

$$m_H \cong 2m_t \cong 262 \text{ GeV}. \tag{5}$$

The first sum rule is essentially a consequence of the relation between the effective Yukawa coupling constants of quarks and leptons and the effective gauge coupling constant of $W^{\pm}$ derived in the model of the NJL type and, therefore, model dependent. On the other hand, the second sum rule is less model dependent since it can be taken as a consequence of Nambu's "0:1:2 relation" or Nambu's quasisupersymmetry.[5] Since all the recent analyses of the constraints on $m_t$ from the experimental data on $m_Z$, $m_W$, and the neutral currents (and muon lifetime)[6] strongly indicate that our prediction of $m_t \cong 131$ GeV is roughly correct, it seems important to know whether the first sum rule or a similar one can be derived in a less model-dependent way. In this Letter, we shall derive a similar sum rule from the Ward identity of Chanowitz and Ellis[7] for the vertex function of the trace of the energy-momentum tensor and two axial-vector currents, and the partially conserved axial-vector current (PCAC) hypothesis.[8]

Let $A_\mu$ and $\theta_{\mu\nu}$ be the axial-vector current and the energy-momentum tensor, respectively. Then, the vertex function of the trace of the energy-momentum tensor and two axial-vector currents is defined by

$$A_{\mu\nu}(p) = \int d^4x\, d^4y\, e^{ip\cdot(x-y)} \langle T(A_\mu(x)A_\nu(y)\theta_\lambda^\lambda(0))\rangle_0 \tag{6}$$

and the vacuum polarization tensor of the axial-vector currents is defined by

$$B_{\mu\nu}(p) = i\int d^4x\, e^{ip\cdot x}\langle T(A_\mu(x)A_\nu(0))\rangle_0 . \tag{7}$$

These are related to each other by the following partially conserved-dilation-current (PCDC) Ward identity derived by Chanowitz and Ellis in 1973:[7]

$$A_{\mu\nu}(p)=\left[2-p\cdot\frac{\partial}{\partial p}\right]B_{\mu\nu}(p)-(p_\mu p_\nu-g_{\mu\nu}p^2)\frac{1}{6\pi^2}\sum_f 1-g_{\mu\nu}\frac{1}{\pi^2}\sum_f m_f^2\,, \tag{8}$$

where the summations are over the constituent fermions (quarks and/or leptons) of which the axial-vector current consists. Define the spectral functions $\Pi_A(p^2)$ and $\Pi_P(p^2)$ by

$$B_{\mu\nu}(p)=(p_\mu p_\nu-g_{\mu\nu}p^2)\Pi_A(p^2)+p_\mu p_\nu\Pi_P(p^2)\,. \tag{9}$$

Then, the Ward identity (8) can be written in the form

$$A_{\mu\nu}(p)=(p_\mu p_\nu-g_{\mu\nu}p^2)\left[-2p^2\frac{\partial}{\partial p^2}\right]\Pi_A(p^2)+p_\mu p_\nu\left[-2p^2\frac{\partial}{\partial p^2}\right]\Pi_P(p^2)-(p_\mu p_\nu-g_{\mu\nu}p^2)\frac{1}{6\pi^2}\sum_f 1-g_{\mu\nu}\frac{1}{\pi^2}\sum_f m_f^2\,. \tag{10}$$

From this follows the identity of

$$p^\mu p^\nu A_{\mu\nu}(p)=-2(p^2)^2\frac{\partial}{\partial p^2}\Pi_P(p^2)-\frac{p^2}{\pi^2}\sum_f m_f^2\,. \tag{11}$$

On the other hand, from the definition (6) follows the identity of

$$\begin{aligned}p^\mu p^\nu A_{\mu\nu}(p)=\int d^4x\,d^4y\,e^{ip\cdot(x-y)}\{&\langle T(\partial^\mu A_\mu(x)\partial^\nu A_\nu(y)\theta_\lambda^\lambda(0))\rangle_0+\langle[A_0(y),[A_0(x),\theta_\lambda^\lambda(0)]\delta(x_0)]\delta(y_0-x_0)\rangle_0\\ &+\langle T([A_0(y),\partial^\mu A_\mu(x)]\delta(y_0-x_0)\theta_\lambda^\lambda(0))\rangle_0+\langle T([A_0(x),\theta_\lambda^\lambda(0)]\delta(x_0)\partial^\nu A_\nu(y))\rangle_0\\ &+\langle T([A_0(y),\theta_\lambda^\lambda(0)]\delta(y_0)\partial^\mu A_\mu(x))\rangle_0\}\,.\end{aligned} \tag{12}$$

Take $A_\mu$ as the axial-vector current consisting of the up and down quarks only and remember the usual PCAC hypothesis,

$$\partial^\mu A_\mu^\pi\cong f_\pi m_\pi^2\phi_\pi\,, \tag{13}$$

where $f_\pi$, $m_\pi$, and $\phi_\pi$ are the pion decay constant, the pion mass, and the pion field, respectively. Then, we can find that $\Pi_P(p^2)$ in the first term of the right-hand side (rhs) of (11) is dominated by the pion pole as

$$\Pi_P(p^2)\cong -f_\pi^2/(p^2-m_\pi^2) \tag{14}$$

and that the first term in the rhs of (12) is dominated by the pion double pole as

$$\int d^4x\,d^4y\,e^{ip\cdot(x-y)}\langle T(\partial^\mu A_\mu(x)\partial^\nu A_\nu(y)\theta_\lambda^\lambda(0))\rangle_0 \cong -2p^2[f_\pi m_\pi^2/(p^2-m_\pi^2)]^2\,. \tag{15}$$

By assuming the dominance of these terms in combining the identities (11) and (12), we obtain the approximate equation

$$2\left[\frac{f_\pi m_\pi^2}{p^2-m_\pi^2}\right]^2\cong 2\left[\frac{f_\pi p^2}{p^2-m_\pi^2}\right]^2+\frac{1}{\pi^2}\sum_{f=u,d}m_f^2\,. \tag{16}$$

In the soft pion limit of $p^2=0$, this equation leads to the approximate sum rule

$$\sum_{f=u,d}m_f^2\cong 2\pi^2 f_\pi^2\,. \tag{17}$$

This sum rule is not satisfied by the current quark masses ($m_u\cong 4.5\pm 1.4$ MeV and $m_d=7.9\pm 2.4$ MeV)[9] but saturated by the constituent quark masses ($m_u\cong m_d \cong 300$ MeV) since $f_\pi\cong 93$ MeV. Therefore, we must take this sum rule as a relation between the constituent quark masses and the pion decay constant, both of which are dynamical quantities closely related to the dynamical scale $\Lambda_c$ ($\cong$100–300 MeV) in quantum chromodynamics.

Although the above sum rule is instructive, it is still far from being similar to the sum rule (2). In order to get closer to the latter, take $A_\mu$ as the axial-vector current consisting of not only all quarks but all leptons or, in other words, the axial-vector part of the GSW weak-isospin $SU(2)_L$ current. In the GWS theory, the equation of motion for $W^-$ is given by

$$V_\mu^+-A_\mu^+=-\frac{2\sqrt{2}}{g}\left[\partial^\nu(\partial_\nu W_\mu^- -\partial_\mu W_\nu^-)+\left(\frac{gv}{2}\right)^2 W_\mu^-\right]+\frac{v}{\sqrt{2}}\partial_\mu\xi^-+\cdots\,, \tag{18}$$

where $V_\mu^+$ and $A_\mu^+$ are the charged vector and axial-vector currents and $\xi^-$ is the charged Higgs scalar component defined by the unitary-gauge representation of

$$\phi=e^{i\tau\cdot\xi/2v}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H\end{pmatrix}\,. \tag{19}$$

Equation (18) seems to indicate that $\xi^-$ may dominate the axial-vector current $A_\mu^+$ and that it may play a role of the pion in the PCAC hypothesis. However, because of its masslessness, $\xi^-$ gives an equal contribution of $-2(\sqrt{2}v)^2p^2$ to the first term in the rhs of (11) and to that in the rhs of (12). Therefore, the identity (11) would trivially hold if $\sum_f m_f^2=0$, which is not acceptable.

A way out of this dilemma is to suppose that the GSW axial-vector current is dominated by a massive (pseudo)scalar ($\Pi$) such as the composite Higgs scalar[3] or the technipion[10] as

$$A_\mu \cong F_\Pi \partial_\mu \Pi . \qquad (20)$$

By following the instructive derivation of the sum rule (17), we obtain the approximate sum rule

$$\sum_f m_f^2 \cong 2\pi^2 F_\Pi^2 , \qquad (21)$$

where the summation is over all quarks and leptons (including the techniquarks or subquarks, if any). In the techniquark (or subquark) model where $F_\Pi=\sqrt{2}v$,[10] the sum rule would predict that the masses of the techniquarks (or subquarks) ($U$'s and $D$'s or $w_1$'s and $w_2$'s) should satisfy the sum rule

$$\sum_{f=U,D} m_f^2 \cong 4\pi^2 v^2 . \qquad (22)$$

This indicates that the average mass of techniquarks (or subquarks) is given by $\langle m_{U,D}^2\rangle^{1/2} \cong \pi v$, $\sqrt{2}\pi v/\sqrt{3}$, $\pi v/\sqrt{2}, \ldots \cong 773, 631, 547$ GeV, ..., depending on the technicolor (or subcolor) gauge group of SU(2), SU(3), SU(4), ....

The sum rule (21) is still very different from (2) since the former indicates $m_f \sim v$ while the latter indicates $m_f \sim G_f v \sim gv$. In other words, if these sum rules coexisted, they would produce a strange situation in which the Yukawa coupling constant or the gauge coupling constant might be determined almost *a priori* in the model as

$$g^2 = 6\pi^2/N_g , \qquad (23)$$

although this relation disagrees with the experimental value of $g^2=4\pi\alpha/\sin^2\theta_W \cong 0.40$ (where $\alpha$ is the fine-structure constant). This situation looks similar to the one in which the vector-meson-dominance coupling constant ($f_\rho$) is determined from the Ward identity for the vertex function of the trace of the energy-momentum tensor and two electromagnetic currents[7] as $g_\rho^2=12\pi^2/\sum_q Q_q^2$, where the $Q_q$'s are the quark charges.[11] Remember also that $f_\rho$ is determined similarly as $g_\rho^2=8\pi^2/e$ (where $e$ is the base of natural logarithm) in the QCD sum rule of Shifman, Vainshtein, and Zakharov.[12] Both of these latter relations agree well with the experimental value of $g_\rho^2/4\pi \cong 2.2\pm 0.3$.

More realistically, the GSW axial-vector current of all quarks and leptons is not dominated by a single pole such as the pion or the technipion but also modulated by many other poles such as the other pseudoscalar mesons ($D_s$, etc.) and the axial-vector mesons ($f_1$, etc.) and by the continuum. Therefore, a sum rule of the type (21) must actually be modified into the more complicated form

$$\sum_f m_f^2 = 2\pi^2(f_\pi^2 + f_{D_s}^2 + \cdots + F_\Pi^2) + \cdots , \qquad (24)$$

so that a strong constraint on quark and lepton masses may not be obtained unless better knowledge of the rhs of this equation is acquired experimentally. However, there may then appear a better chance for the coexistence of the sum rule (2) and the sum rule (24). If the top quark and the Higgs scalar are found with masses of about 130 and 260 GeV as predicted from the sum rules (2) and (3), respectively, this will tell us that not only the gauge bosons but also the Higgs scalars may be taken as collective excitations of quarks and leptons as in the unified model of the NJL type.[3] If this is the case, in order to be consistent with the composite picture of not only quarks and leptons but also the gauge bosons and Higgs scalars, we must adopt the principle of triplicity[13] in which a certain physical quantity such as the weak current can be taken equally well as either one of a composite operator of hadrons, that of quarks and that of subquarks.

In conclusion, I must add the following comments: Recently, several authors[14-16] have closely investigated the possibility of top-quark condensation in which the Higgs scalar appears almost as a $\bar{t}t$ bound state. The simple analysis of the NJL model by Miransky, Tanabashi, and Yamawaki[14] is based on the Schwinger-Dyson equation while the more precise and elaborate one of the gauged NJL model by Bardeen, Hill, and Linder[15] is further based on the renormalization-group equations for the full standard model. The latter analysis instructively indicates how the sum rules (2) and (3) are modified not only by the eminence of the top quark but also by the presence of the full interactions. Nambu[16] has presented a new sum rule for the masses of $t$, $H$, $W$, and $Z$ based on "bootstrap symmetry breaking," which is different from either one of (2) or (3). The sum rule (21) is still very distinct from any one of these sum rules since it is derived current algebraically and is strongly dependent on "scalar or pion-pole dominance."

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