Duality Transformation for Non-Abelian Lattice Gauge Theories

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In non-Abelian lattice gauge theories, the Gauss's-law constraint is solved, enabling gauge-invariant local dynamics. For the 2+1, SU(2) case, the result is a quantum theory of a discretized membrane. Further, this is equivalent to a theory of integer-valued scalars with derivative interactions and a "dual" gauge invariance. This is the dual form of 2+1, SU(2) gauge theory.

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It is widely believed[1] that certain topological degrees of freedom (e.g., monopoles) are responsible for screening the massless gluons and confining the quarks in quantum chromodynamics. Such a mechanism is in operation[2] in 2+1 and 3+1 U(1) lattice gauge theories. The 2D XY model[3] has provided valuable hints for this proposal. Duality transformations[4,5] separate such topological degrees of freedom and isolate the effects of compactness of the group. Therefore, an analogous transformation for non-Abelian lattice gauge theories could prove to be valuable. However, at present, the duality transformation has been carried out successfully only for symmetries corresponding to Abelian groups. For non-Abelian groups similar attempts have appeared to be cumbersome. This may partly explain why earlier workers[6] proposed topological objects corresponding to Abelian subgroups of the non-Abelian group as relevant for confinement, even though it is not apparent that this is enough.

We show here, in the context of 2+1, SU(2) lattice gauge theory without matter, that it is possible to explicitly carry out the duality transformation in the Hamiltonian formalism. The dual theory has some features analogous to the Abelian case.[5]

The Hamiltonian[7] is

$$H=\sum_{n,i}E(ni)^2-K\sum_{n,i>j}\mathrm{tr}[U(nij)+U^\dagger(nij)]\,, \qquad (1)$$

where

$$U(nij)=U(ni)U(n+i,j)U^\dagger(n+j,i)U^\dagger(nj)$$

and K is the coupling constant. Here n labels the sites of a 2D square lattice and i is the unit vector along the ith direction. Each link (ni) is associated with an SU(2)-symmetric top, whose configuration (i.e., the rotation matrix from space-fixed to body-fixed axes) is given by an SU(2) matrix $U(ni)$. The angular momenta with respect to space-fixed and body-fixed frames are $E_+(ni)$ and $E_-(n+i,i)$, respectively. They generate left and right SU(2) rotations on $U(ni)$. $E_+(ni)$ and $E_-(n+i,i)$ commute with each other and, moreover, $E_+(ni)^2=E_-(n+i,i)^2$; i.e., the total angular momentum is the same in both frames and this is denoted by $E(ni)^2$ in Eq. (1). However, the axis of rotation can have arbitrary inclinations in the two frames. Complete labeling of the basis requires[8] eigenvalues $m_+(ni)$ and $m_-(n+i,i)$ of both $E^3_+(ni)$ and $E^3_-(n+i,i)$, in addition to the label $j(ni)$ for $E_+(ni)^2=E_-(n+i,i)^2$. Thus a basis for the Hilbert space is $|\{j(ni),\ m_+(ni),\ m_-(n+i,i)\}\rangle$ where all of them are integral multiples of $\frac{1}{2}$ ("half-integers") with $j(ni)\geq|m_+(ni)|$ and $|m_-(n+i,i)|$. This is before the Gauss's-law constraint is implemented.

The non-Abelian Gauss's law giving the physical states is $\sum_i[E_+(ni)+E_-(ni)]=0$ at each site, implying the physical states are invariant under local gauge transformations. This is to be contrasted with the continuum case, where the electric-field strengths $\mathcal{E}^a_i$, $a=1,2,3$, commute with each other, but Gauss's law $D_i(A)^{ab}\mathcal{E}^b_i=0$ involves the vector potential A^a_i which does not commute with $\mathcal{E}^a_i$. On the lattice $-E_-(n+i,i)$ is just the parallel transport of $E_+(ni)$ from n to $n+i$. In the limit of lattice spacing $a\to 0$, $E_+(ni)\to a^{3/2}\mathcal{E}_i(\mathbf{x})$. The advantage of using the basis where $E^3_+(ni)$ and $E^3_-(ni)$ are diagonal is that *Gauss's law can be explicity and locally solved* to get a basis for gauge-invariant states. This has been noted by one of us earlier.[9] For this we make a change to the basis where the following operators involving sums of angular momenta are diagonal,

$$\{E(n1)^2,\ E(n2)^2,\ [E_+(n1)+E_+(n2)]^2,\ [E(n1)+E(n2)]^2,\ [E^3_+(n1)+E^3_+(n2)],\ [E^3_-(n1)+E^3_-(n2)]\}\,,$$

and a further change of basis using

$$\{E(n1)^2,\ E(n2)^2,\ [E_+(n1)+E_+(n2)]^2,\ [E_-(n1)+E_-(n2)]^2,$$
$$[E_+(n1)+E_+(n2)+E_-(n1)+E_-(n2)]^2,\ [E^3_+(n1)+E^3_+(n2)+E^3_-(n1)+E^3_-(n2)]\}\,.$$

The gauge-invariant states are just those where, for each n, the last two operators of the above set have zero eigenvalues. In this case, moreover, the eigenvalues of $[E_+(n1)+E_+(n2)]^2$ and $[E_-(n1)+E_-(n2)]^2$ are equal and we label

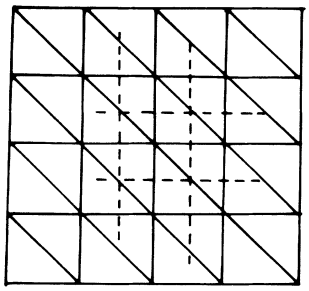

FIG. 1. Dual lattice with one set of diagonals drawn. The original plaquettes are represented by dotted lines.

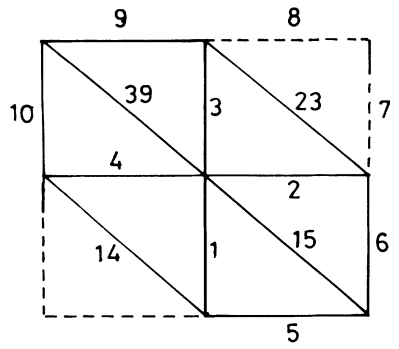

FIG. 2. The effect of the plaquette operator is to change the lengths of the six arms of the corresponding hexagon on the dual lattice.

them by $j(n12)$. Thus a basis for gauge-invariant states is

$$|\{j(n1), j(n2), j(n12)\}\rangle . \tag{2}$$

Thus three half-integers are associated with each site. The number of physical degrees of freedom for gluons is also three: one transverse polarization for each of the three gluons. Such a matching is valid for higher dimensions and other gauge groups also.[9,10]

In the above basis for gauge-invariant states, the labels are not mutually independent. $j(n12)$ is a value obtained by adding angular momenta $j(n1)$ and $j(n2)$ [or also $j(n-1,1)$ and $j(n-2,2)$] and hence the three numbers satisfy triangle inequalities. These constraints may be conveniently represented on the dual lattice. We associate $j(ni)$ $(i=1,2)$ with the link dual to (ni) and $j(n12)$ with the diagonal completing the corresponding dual links into a triangle (Fig. 1). We therefore associate a half-integer with each link of the dual lattice where one set of diagonals is also drawn (Fig. 2). *This means a basis for gauge-invariant states is provided by all triangulations with half-integer sides and with coordination number six* (the Euler characteristic being equal to that for the 2D lattice).

Gauge-invariant local dynamics.—We consider the dynamics on this gauge-invariant subspace directly. *This provides gauge-invariant local dynamics for non-Abelian gauge theories.*

The piece $E(ni)^2$ of the Hamiltonian (1), when acting on basis (2), is simply replaced by $j(ni)\,[j(ni)+1]$. The plaquette term $\mathrm{tr}[U(nij)]$ has the effect of changing the lengths of the six lines emanating from the corresponding dual site n^* (Fig. 1) by $\pm\frac{1}{2}$. The transition-matrix elements from the lengths $\{j_1,j_2,j_3,j_4,j_{15},j_{39}\}$ of these six lines (Fig. 2) to the corresponding primed values can be computed[11] by repeated applications of the Wigner-Eckart theorem. The matrix element depends on the lengths $j_{14},j_{23},j_5,j_6,j_9,j_{10}$ (of the edges of the hexagon), though these do not change under the action of the plaquette centered at n^*. The matrix element is[11]

$$\begin{aligned}\langle M\rangle = &\begin{Bmatrix} j_{14} & j_1 & j_4 \\ \frac{1}{2} & j_4' & j_1' \end{Bmatrix}\begin{Bmatrix} j_{23} & j_2 & j_3 \\ \frac{1}{2} & j_3' & j_2' \end{Bmatrix}\begin{Bmatrix} j_5 & j_1 & j_{15} \\ \frac{1}{2} & j_{15}' & j_1' \end{Bmatrix}\begin{Bmatrix} j_6 & j_2 & j_{15} \\ \frac{1}{2} & j_{15}' & j_2' \end{Bmatrix}\begin{Bmatrix} j_9 & j_3 & j_{39} \\ \frac{1}{2} & j_{39}' & j_3' \end{Bmatrix}\begin{Bmatrix} j_{10} & j_4 & j_{39} \\ \frac{1}{2} & j_{39}' & j_4' \end{Bmatrix} \\ &\times\sqrt{2j_1+1}\sqrt{2j_2+1}\sqrt{2j_3+1}\sqrt{2j_4+1}\sqrt{2j_{15}+1}\sqrt{2j_{39}+1} \\ &\times\sqrt{2j_1'+1}\sqrt{2j_2'+1}\sqrt{2j_3'+1}\sqrt{2j_4'+1}\sqrt{2j_{15}'+1}\sqrt{2j_{39}'+1}(-1)^{j_5+j_{14}+j_{23}-j_9-j_6+j_{10}+j_2-j_2'-j_1+j_1'},\end{aligned} \tag{3}$$

where we have a 6-j symbol (the curly brackets) associated with each of the six triangles of the hexagon. The angular momenta involved in each 6-j symbol are the original (j) and the new lengths (j') of the two arms of the corresponding triangle, the edge of the hexagon, and $\frac{1}{2}$ corresponding to the tensorial property of $U(ni)$.

Thus the time evolution involves local fluctuations of the triangulated surface. We have therefore *mapped 2+1, SU(2) lattice gauge theory to a specific quantum dynamics of a discretized membrane.*

Duality transformation.—Triangle-inequality constraints in (2) are naturally untangled by assigning elements φ of a metric space X to the vertices such that the distance $d(\varphi_1,\varphi_2)$ in X between φ_1 and φ_2 associated with the two vertices of a dual link is the corresponding j. However, it is not possible to have global isometric embedding of a triangulated surface in any metric space. A simple counterexample is provided by the hexagon (Fig. 2) where all six arms and one side have a common length $j>0$ whereas the other five sides are zero. The triangle inequality is valid for each of the six triangles. However, one side of the hexagon is longer than the sum of the other five sides, precluding its embedding in any metric space.

On the other hand, each triangle can be separately embedded in a metric space (local isometric embedding). For the metric space, we may choose a square lattice

equipped with the taxicab metric where the distance between nearest neighbors is $\frac{1}{2}$. We thus associate two integers to each vertex of the triangle so as to reproduce the length of its sides. The assignment is not unique. Any two choices will be related by a lattice translation and a "lattice rotation," once we presume an orientation for the triangle. By a lattice rotation about a point $\mathbf{n}$ we mean the following. A point $\mathbf{m}\neq\mathbf{n}$ moves to $\mathbf{m}'$ such that the distance from $\mathbf{n}$ is unaltered. All other points l move to l' such that $d(l,\mathbf{n})=d(l',\mathbf{n})$, and $d(l,\mathbf{m})=d(l',\mathbf{m}')$. This defines a rotation operation for almost all lattice points.

Thus each site n^* of the dual lattice is associated with twelve ordered integers $\varphi(n^*)$ corresponding to the six vertices of the triangles around n^*. These coordinates are not all independent because the common edge of two adjacent triangles should have the same length. This provides three constraints among the twelve coordinates for each dual site n^*. The effect of the plaquette operator in the Hamiltonian (1) corresponds to a hopping to a nearest neighbor of the 2D metric space for each vertex, without violating the above constraint. As a result there are $12-3=9$ independent vibrational modes. If the discreteness of these coordinates is ignored, these vibrational modes are massless because the coupling involves only the differences in coordinates of the neighbors.

Not all of these nine modes are physical. Different assignments of the twelve coordinates reproduce the same triangulations and are to be interpreted as gauge-equivalent states with respect to a new local gauge invariance (which will be referred to as the dual gauge invariance g^*). A dual gauge transformation corresponds to a lattice translation and a lattice rotation of coordinates associated with any triangle. The scalar field $\{\varphi(n^*)\}$ is the dual potentials in the sense that they are the potentials defined using the electric fields instead of the magnetic fields.

The dual gauge transformation has three parameters for each triangle, each triangle is shared by three vertices, and to each vertex there are six triangles. As a result, six of the nine vibrational modes are unphysical. This leaves three physical modes corresponding to the three transverse gluons of the 2+1, SU(2) theory. The situation is analogous to 2+1, U(1) lattice gauge theory which is mapped[5] onto a theory with one integer-valued scalar. As in that case, we may replace[10] the discrete-valued φ by a field taking continuous values in $(-\infty, +\infty)$ together with certain "topological" degrees of freedom. Their implication for linear confinement and the continuum limit requires more detailed analysis, and will be presented elsewhere.[11]

To summarize, we have solved the Gauss's-law constraint for non-Abelian lattice gauge theory locally and explicitly and obtained dynamics on the physical states. This dynamics is local and the 2+1, SU(2) case is interpreted to be a quantum theory of a discretized membrane. This allows a close relationship between the 2+1, SU(2) gauge theory and 2+1 gravity. We have further mapped this theory onto a theory of integer-valued scalars with nonlinear derivative interactions and a new (dual) gauge invariance. This scalar field is the dual potential. This theory is analogous to the picture of photons plus monopole plasma of 2+1, U(1) lattice gauge theory and is to be regarded as the dual transform of the non-Abelian lattice gauge theory. All effects of the compactness of the link variables $U(ni)$ and of the gauge group have been completely isolated in our approach.

The above ideas can be extended to higher dimensions and other gauge groups.[10] It is yet to be seen whether this approach solves the $N\rightarrow\infty$ limit of SU(N) gauge theories. Matter fields in interactions with the gauge bosons can be incorporated essentially in the same way as presented here.[10] It is to be hoped that the confinement mechanism will be unambiguously settled and nonperturbative effects can be handled. These techniques provide the dual transformation of other theories with (global or local) non-Abelian symmetries as well.

[1]G. 't Hooft, in *Proceedings of the European Physical Society International Conference on High Energy Physics, Palermo, Italy, 1975,* edited by A. Zichichi (Editrice Compositori, Bologna, 1976); Phys. Scr. **20**, 133 (1982); S. Mandelstam, in *Extended Systems in Field Theory,* Proceedings of the Meeting at Ecole Normale Superieure, Paris, 1975, edited by J. L. Gervais and A. Neveu [Phys. Rep. **23**, 237 (1976)]; in *Monopoles in Quantum Field Theory,* edited by N. S. Craigie, P. Goddard, and W. Nahm (World Scientific, Singapore, 1982).

[2]A. M. Polyakov, Phys. Lett. **59B**, 82 (1975); Nucl. Phys. **B120**, 429 (1977).

[3]J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973); J. Phys. C **7**, 1046 (1974).

[4]J. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).

[5]T. Banks, J. Kogut, and R. Myerson, Nucl. Phys. **B129**, 493 (1977).

[6]G. 't Hooft, Nucl. Phys. **B138**, 1 (1978); **B190**, 455 (1981).

[7]J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).

[8]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977), 3rd ed.

[9]H. S. Sharatchandra, Nucl. Phys. **B196**, 62 (1982).

[10]Ramesh Anishetty, Manu Mathur, and H. S. Sharatchandra (to be published).

[11]Ramesh Anishetty and H. S. Sharatchandra (to be published).