Models of Intermittency in Hydrodynamic Turbulence

Robert H. Kraichnan
303 Potrillo Drive, Los Alamos, New Mexico 87544
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A heuristic model for evolution of the probability distribution (PDF) of transverse velocity gradient s in incompressible Navier-Stokes turbulence is distilled from an analytical closure for Burgers turbulence. At all Reynolds numbers $\mathcal{R}$, the evolved PDF is $\propto |s|^{-1/2}\exp(-\text{const}\times|s|/\langle s^2\rangle^{1/2})$ for large $|s|$. The model suggests that skewnesses and flatnesses are asymptotically independent of $\mathcal{R}$, and that cascade to smaller scales is not a fractal process. For Burgers dynamics, both simulations and the analytical closure give a PDF $\propto |\xi|^{-1}\exp(-\text{const}\times|\xi|/\langle\xi^2\rangle^{1/2})$ for large negative velocity gradient ξ.

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The Kolmogorov theory of the inertial range (K41) is fifty years old,[1] but fundamental questions about the small scales of Navier-Stokes turbulence remain unresolved. Kolmogorov postulated a self-similar cascade of energy to smaller scales (higher wave numbers). Implications include an energy spectrum function of the form $E(k)\propto k^{-5/3}$ and a dissipation wave number $k_s=(\epsilon/\nu^3)^{1/4}$, where ϵ is the mean rate of energy dissipation per unit mass and ν is kinematic viscosity.

K41 rests on two distinct assumptions: First, that energy transfer to small scales proceeds by stepwise cascade, local in wave number; second, that information is lost in the cascade, so that the normalized statistics of velocity fields band limited to each, say, decade band of wave numbers in the inertial range ($k_0 \ll k \ll k_s$) are identical and determined soley by ϵ.

The first assumption is very solid. It is difficult to cook up statistics in which it fails, provided that the inertial-range spectrum exponent is $-1>n>-3$. In particular, the assumption is easily validated in two very different cases: the initial transfer in an initially Gaussian field with the K41 value $n=-\frac{5}{3}$; and transfer in the highly coherent, sawtooth solutions of Burgers' equation,[2] where $n=-2$. The second assumption is much less secure. It has been challenged by Landau and Lifshitz,[3] Kolmogorov himself,[4] and others. Their arguments have led to fractal-cascade models in which the statistics of successive band-limited velocity fields become ever more intermittent until viscosity intervenes. In the limit of infinite Reynolds number, such models confine the dissipation to a vanishingly small fraction of the entire flow volume. They yield spectrum exponents $n<-\frac{5}{3}$. References 5-14 are taken from a large and varied literature.

The models to be described here have a different origin and lead to a different picture of intermittency. They are based on analytical approximations ("mapping closures")[15,16] for the probability distribution function (PDF) of velocity gradient. The competition between viscous relaxation and the straining process that produces small scales is followed in x space. Neither the second K41 assumption nor its fractal variants are supported. The PDF of velocity gradient is linked to the Gaussian statistics of large scales. Initial statistics are not forgotten during straining. The model dynamics are not fractal but some predictions may mimic those of fractal dynamics. In a different spirit, Castaing recently has represented intermittency by superposition of Gaussian PDF's.[11,13]

Mapping closures are based on the distortion of a Gaussian reference field in x space into a dynamically evolving non-Gaussian field. The distortion involves locally determined changes of field amplitude and gradient. The mapping functions are nonstochastic in the simplest formulations. They are determined self-consistently so that the evolution of certain one-point PDF's under the mapping matches the evolution under the dynamical equations.

The Navier-Stokes (NS) equation yields

$$\mathcal{D}u_{i,j}/\mathcal{D}t=-u_{i,m}u_{m,j}-p_{,ij}+\nu\nabla^2 u_{i,j}\,, \tag{1}$$

where $\mathbf{u}$ is the velocity field, $\mathcal{D}/\mathcal{D}t\equiv\partial/\partial t+\mathbf{u}\cdot\nabla$, $u_{i,j}\equiv\partial u_i/\partial x_j$, p is the pressure per density, and ν is the kinematic viscosity. Let s denote any transverse component of $u_{i,j}$ and suppose that s has a Gaussianly distributed initial value s_0. The simplest heuristic form of mapping closure makes s evolve through the action of a nonstochastic effective stretching function $J(s_0,t)$ that obeys a nonlinear equation of motion:

$$s=J(s_0,t)s_0\,, \tag{2}$$

$$\partial J/\partial t=|s_0|J^2-\nu k_d^2 J^3\,. \tag{3}$$

The first term on the right-hand side of (3) comes from the quadratically nonlinear straining term in (1). Alone, this term would induce extreme intermittency in s and a singularity in finite time. The second term on the right-hand side represents viscous decay of $s=s_0J$ with decay constant proportional to the square J^2 of the stretching ratio and thereby to the square of the effective intensification ratio of velocity gradients of local flow structures. Here k_d is a characteristic dissipation wave number for s_0 fluctuations, and the ratio $\langle s_0^2\rangle^{1/2}/\nu k_d^2$ is an effective initial Reynolds number, if the initial spectrum is compact. The only other parameter is the normalized evolution time $\langle s_0^2\rangle^{1/2}t$. Decay of velocity amplitudes is ignored, so the model is properly applicable only over times short compared to overall decay times. Decay effects are included in the analytical closure (16) for Burgers turbulence that suggested (2) and (3). Random forcing that yields steady states is easily added.

At large enough values of $|s_0|$, (3) quickly leads to a near equilibrium in which J grows to make the two terms on the right-hand side balance:

$$J \approx |s_0|/\nu k_d^2, \quad |s| \approx s_0^2/\nu k_d^2, \quad J \approx |s|^{1/2}/(\nu k_d^2)^{1/2}. \tag{4}$$

The PDF of s is

$$P(s) = P_0(s_0)\partial s_0/\partial s\,, \tag{5}$$

where

$$P_0(s_0) = (2\pi\langle s_0^2\rangle)^{-1/2}\exp(-\tfrac{1}{2}s_0^2/\langle s_0^2\rangle)\,. \tag{6}$$

Equations (4)–(6) imply that $P(s)$ at large enough $|s|$ has the form

$$P(s) \approx \left(\frac{\nu k_d^2}{8\pi\langle s_0^2\rangle|s|}\right)^{1/2}\exp\left(-\frac{\frac{1}{2}\nu k_d^2|s|}{\langle s_0^2\rangle}\right) \quad \left(|s| \gg \frac{\langle s_0^2\rangle}{\nu k_d^2}\right). \tag{7}$$

Equation (7) recalls the near-exponential PDF tails observed in incompressible NS simulations[9–14] and elsewhere.[17] Figure 1 shows the PDF (5), evolved at $t=0.3$ under (2) and (3) from a start at $t=0$ with $s=s_0$, $\langle s_0^2\rangle = 1$, $\nu k_d^2 = 1.4$. This curve is startlingly close to the superimposed PDF of $\partial u_1/\partial x_2$ measured by Vincent and Meneguzzi[14] in an isotropic turbulence simulation at Taylor microscale Reynolds number $\mathcal{R}_\lambda \approx 150$. The upward flare of the skirt of the simulation PDF is accurately reproduced. It is notable that this match is obtained at model parameter values that give initial Reynolds number and normalized evolution time both of order unity.

An initial spectrum shape enters the closure only through the evolution time for which it is valid to neglect overall decay of small scales. Thus Fig. 1 could represent a stage of evolution of an initial Gaussian field with spectrum of K41 form and $k_d \approx k_s$. The good match to a high-$\mathcal{R}_\lambda$ simulation arises in essentially one characteristic dissipation-range eddy-circulation time, without benefit of buildup of intermittency in an inertial-range cascade. For given k_d and ν, increase of macroscale Reynolds number $\mathcal{R}$ slows overall decay of small scales. The tails at very large $|s|/\langle s_0^2\rangle^{1/2}$ form quickly and thereby are independent of $\mathcal{R}$. The steady-state solution of (3), which is (7) for all s, describes an equilibrium at

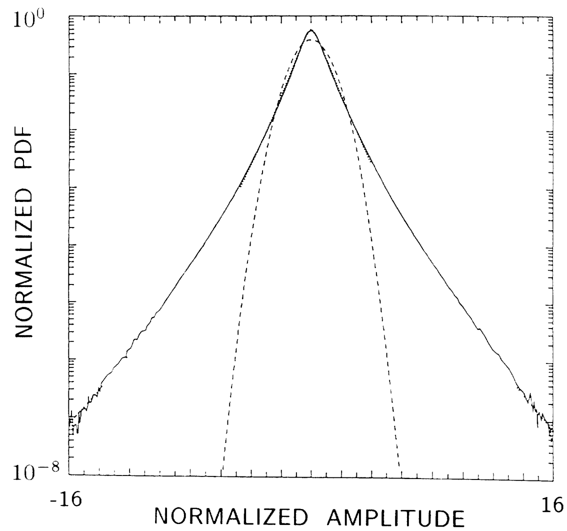

FIG. 1. PDF's, normalized by standard deviation, for s evolved under (2) and (3) (dotted line), $\partial u_1/\partial x_2$ from a simulation at $\mathcal{R}_\lambda \approx 150$ (solid line) (Ref. 14), and a Gaussian process (dashed line).

$\mathcal{R} = \infty$. The kurtosis $K_s = \langle s^4\rangle/\langle s^2\rangle^2$ is 35/3 for this limiting solution. Then realizability inequalities require that the skewness of s also be finite at $\mathcal{R} = \infty$. This implies an inertial-range exponent $n = -\frac{5}{3}$: Localness of energy transfer ties values $n < -\frac{5}{3}$, associated with fractal models, to skewnesses that increase as powers of Reynolds number.[6] The present closure suggests that the $\mathcal{R}$ dependence of K_s resides not in the skirts of the PDF but in the deviations from (7) near the maximum of the PDF.

The value 35/3 is as large as most flatnesses measured at high $\mathcal{R}_\lambda$. Some atmospheric studies yield higher values,[7] but with unknown effects from spottiness of large scales. A finite limiting value for K_s does not seem ruled out, given Gaussian large scales. Qian has inferred such a limit from a bold, and uncontrolled, cumulant expansion.[18]

The PDF of two-point velocity difference must change smoothly from Gaussian, at very large separations $\Delta\mathbf{x}$, to that of s as $|\Delta\mathbf{x}| \to 0$. Mapping closure also determines multipoint statistics, such as this $\Delta\mathbf{x}$ dependence, as described after the Burgers analysis.

The closure (2),(3) was inspired by an analytic closure for Burgers's equation

$$\mathcal{D}u/\mathcal{D}t = \nu u_{xx}, \quad \mathcal{D}\xi/\mathcal{D}t = -\xi^2 + \nu\xi_{xx}\,, \tag{8}$$

where $u(x,t)$ is the velocity and $\xi \equiv u_x \equiv \partial u/\partial x$.

The dominant behavior under Burgers's equation is steepening of negative velocity gradients into viscosity-limited shocks. During this process an initially multivariate-Gaussian field u retains a nearly Gaussian-univariate distribution $P(u)$ while the univariate distribution $Q(\xi)$ of ξ becomes highly intermittent, even at low Reynolds number.

Analytical mapping closure starts with a multivariate-Gaussian *reference field* $u_0(z)$. It is assumed that the actual field $u(x,t)$ and laboratory position x are related to u_0 and the reference coordinate z by[15,16]

$$u=X(u_0,t),\quad dz/dx=J(u_0,\xi_0,t)\,, \tag{9}$$

where X and J are ordinary, nonstochastic functions and the arguments u_0 and ξ_0 are values measured at any point z. Thus

$$\xi=\xi_0 J(u_0,\xi_0)\partial X/\partial u_0 \equiv Y(u_0,\xi_0)\,, \tag{10}$$

where $\xi_0 \equiv \partial u_0/\partial z$.

The analysis is simplified by the approximation that $P(u)$ is Gaussian and that u and ξ are statistically independent. Therefore I assume that the single-point joint PDF of u and ξ has the form

$$P(u,\xi)=P(u)Q(\xi)\,, \tag{11}$$

$$u=X(u_0,t)=r(t)u_0,\quad \xi=r(t)\xi_0 J(\xi_0)\equiv Y(\xi_0,t)\,. \tag{12}$$

Here $r(t)$ measures the decay of velocity amplitudes under viscosity. Energy conservation then yields

$$dr/dt=-\nu r\langle\xi^2\rangle/\langle u^2\rangle\,. \tag{13}$$

It is easily shown[15,16] that $Q(\xi)$ exactly obeys the reduced Liouville equation

$$\frac{\partial Q(\xi)}{\partial t}+\frac{\partial}{\partial \xi}\left(\left[\frac{\mathcal{D}\xi}{\mathcal{D}t}\right]_{C:\xi}Q(\xi)\right)=\xi Q(\xi)\,, \tag{14}$$

where $[\,]_{C:\xi}$ denotes ensemble mean conditional only on a given value ξ and the divergence term on the right-hand side expresses the difference of measure between Lagrangian and Eulerian coordinates. On the other hand, (9)-(12) give

$$Q(\xi)=Q_0(\xi_0)(\partial Y/\partial\xi_0)^{-1}N/J\,, \tag{15}$$

where $Q_0(\xi_0)$ is the Gaussian PDF of the gradient of the reference field, the factor $1/J$ expresses the change of measure associated with squeezing or stretching of z to give x, and $N(t)$ normalizes $Q(\xi)$ to unity.[19] The requirement that (15) give the same $Q(\xi)$ as (14) leads to an evolution equation for J:[16]

$$\frac{\partial J}{\partial t}=-r\xi_0 J^2-\frac{1}{r\xi_0 Q(\xi)}\int_{-\infty}^{\xi}[\alpha(\xi')-\xi']Q(\xi')d\xi' +\nu(r\xi_0)^{-1}[\xi_{xx}]_{C:\xi}+\nu J\langle(\xi_0 J)^2\rangle/\langle u_0^2\rangle\,, \tag{16}$$

where $\alpha(\xi)=\partial\ln(N/J)/\partial t$.

To obtain closure, $[\xi_{xx}]_{C:\xi}$ must be evaluated. Given (12), this can be done exactly by differentiating (12) and using the known statistics of the reference field.[15,16] The result is

$$[\xi_{xx}]_{C:\xi}=-r\xi_0 k_d^2\left[J^3+\frac{\xi_0}{3}\frac{\partial J^3}{\partial\xi_0}\right] +rC_2\left[\frac{\partial J^3}{\partial\xi_0}+\frac{\xi_0}{2}J\frac{\partial^2 J^2}{\partial\xi_0^2}\right]\,, \tag{17}$$

with $C_2=\langle(\partial\xi_0/\partial z)^2\rangle$, $k_d^2=C_2/\langle\xi_0^2\rangle$.

The first term on the right-hand side of (16) and the first term on the right-hand side of (17) are like the terms on the right-hand side of (3). The J^2 term in (16) comes from the $-\xi^2$ term in (8). The derivative terms on the right-hand side of (17) come from exact treatment of ξ_{xx} under the space-varying distortion J. The integral term in (16) arises from the effect on $P(u,\xi)$ of the N/J factor in (15); it makes (16) an integro-differential equation that must be solved iteratively. The derivative and integral terms [ignored in (3)] play an essential role in shaping $Q(\xi)$ near its maximum. J at large negative ξ is controlled by the terms like those in (3), with the result that a nearly exponential tail is also a property of $Q(\xi)$, but only for negative ξ. The measure factor $1/J$ in (15) makes the prefactor take the form $|\xi|^{-1}$, instead of the $|s|^{-1/2}$ in (7). No factor $1/J$ is included in (5) because the NS dynamics are incompressible.[20]

$P(u,\xi)$ is unchanged if the transformation $r(t)$, $J(\xi_0,t)$ is replaced by three successive measure-preserving operations on initial field realizations: (a) an effective viscous relaxation of the reference field that changes $\langle\xi_0^2\rangle/\langle u_0^2\rangle$ by the factor $[N(t)]^2$ and $\langle u_0^2\rangle$ by a related factor $[r_0(t)]^2$; (b) a squeezing of the relaxed reference field by the factor $J_N(\xi_0,t)=J(\xi_0,t)N(t)$; (c) a reduction of amplitudes by the factor $r(t)/r_0(t)$.

Figure 2 compares the $Q(\xi)$ obtained from the solution of (11)-(13) and (15)-(17) with direct simulations of (8).[16] The latter were started from a multivariate-Gaussian distribution of $u(x,t=0)=u_0$ with an energy spectrum $\propto k^2\exp(-k^2/k_0^2)$. Each simulation involved 10^5 points unit spaced on a cyclic line segment. Ten runs were made with parameter assignments $\nu=2$, $\langle u_0^2\rangle=1$, $k_0=0.1$, which give $\langle\xi_0^2\rangle=0.015$, $C_2=3.75\times10^{-4}$ and Reynolds number $\mathcal{R}=\langle u_0^2\rangle^{1/2}k_0^{-1}/\nu=5$. The same values were used in integrations of the closure equations. The departure from Gaussian shape is still increasing at $t=20$. The flare of the skirt is reminiscent of Fig. 1. Nothing is adjustable in the closure.

Burgers and NS dynamics are markedly different under mapping closure as $\mathcal{R}\to\infty$. The sharpening of shocks as $\nu\to 0$ implies that $K_\xi=\langle\xi^4\rangle/\langle\xi^2\rangle^2$ increases indefinitely with $\mathcal{R}$. This is reflected in the analytical form of the closure PDF: The prefactor $|\xi|^{-1}$ makes $Q(\xi)$ non-normalizable if the $|\xi|\to\infty$ form is extended to $\xi=0$. Both closure solution and simulations give increasingly sharp peaks at the maximum of $Q(\xi)$ as the

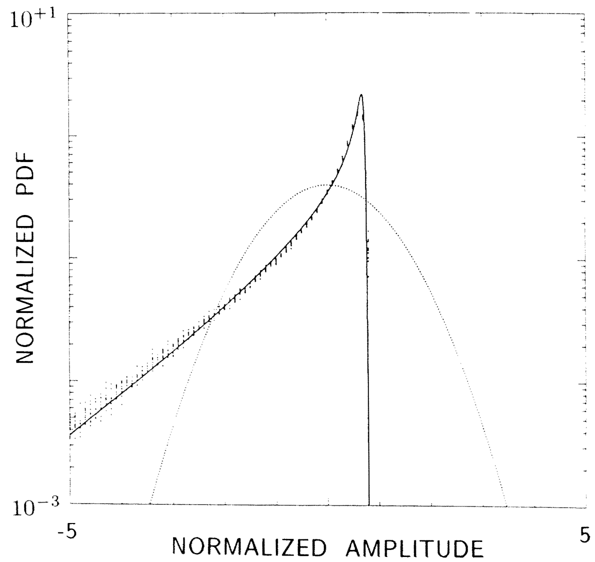

FIG. 2. Normalized PDF of $u_x = \xi$ for Burgers turbulence according to simulations (data points) and mapping closure (solid line) at time $t = 20$.

Reynolds number increases.

Mapping closures evolve each realization in the initial Gaussian ensemble so that multipoint statistics, and spectra, may be computed over the evolved ensemble. In the Burgers case at large $\mathcal{R}$, sharpening of negative gradients into viscosity-limited shocks is exhibited by the model realizations, and the inertial-range spectrum exponent is $n = -2$.

Analytical mapping closures for NS dynamics must deal explicitly with the pressure term in (1). A prime challenge is the sharp difference in statistics between two and three dimensions. Velocity vector potentials and effective pressure fields that are time-dependent functions of local reference-field velocity and velocity gradient are tools for making mappings that properly express the solenoidal straining of individual realizations. Will the predictions from (2) and (3) survive a proper treatment of pressure in three dimensions and a consequently less deterministic evolution of strain? A relevant finding is that the PDF tail is unchanged in form if a rapidly varying factor $c(t)$ is inserted in the first term on the right-hand side of (3).

The present results from a crude closure do not force me to conclude that fractal processes and corrections to K41 are absent in homogeneous turbulence. But many high-$\mathcal{R}_\lambda$ data can be explained without fractals. The high-vorticity regions in isotropic simulations[10,12,14] are ropes of vorticity without obvious fractal structure. Fractal structure may yet show at $\mathcal{R}_\lambda$ values that now are out of reach.

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[19] The $1/J$ weighting was overlooked in Ref. 15. I am grateful to S. B. Pope, who independently pointed out the need for this factor.

[20] This argument is simplistic, but it seems justified at the crude level of (3).