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Eigenvalue Statistics of Distorted Random Matrices

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A numerical study of a set of random matrices which interpolate Poisson and Gaussian orthogonal ensembles is reported. The result indicates that the transition from Poisson to Wigner distributions for the nearest-level spacing does not depend on the details of the random-matrix parametrization and is essentially governed by a single parameter. Brody's one-parameter interpolation formula is found to describe the transition rather well.

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The random-matrix theory of Dyson and Mehta[1] attracted renewed attention in recent years as the study of chaotic motion in quantum system progressed. The Wigner distribution for the nearest-neighbor level spacing, which is predicted from the random-matrix theory, is widely regarded as a prime candidate for the quantum signature of chaotic dynamics.[2,3] A number of studies indicate that the integrable system, on the other hand, shows Poisson level-spacing distribution when quantized.[4,5] The transition between these two extremes has been the subject of several works.[6-10] Although there are some conjectures on "universality" in the transition of level-spacing distribution from Poisson to Wigner form, we have yet to identify the suitable order parameter to describe such a transition. Only after having quantitative description for such a transition, we should be able to discuss its possible relation to the transition from integrable to chaotic motion in the classical counterpart.

In this Letter, we attempt to quantify the description of quantum level statistics. To that end, we perform numerical experiments of two sets of random matrices which interpolate the diagonal random matrix and the invariant orthogonal random matrix. We look at the moments of the nearest-level-spacing distribution obtained by diagonalizing our model random matrices.

The result of this numerical study seems to indicate that there is indeed a class of random matrices which shows a single pattern for the shape change from Poisson to Wigner distribution, and a single parameter, which we identify as the "effective rank" of matrices, controls the transition. Curiously enough, the one-parameter interpolation formula of Brody[11] is found to describe this universal transition rather well, at least good enough for the typical statistical error encountered in both numerical calculations and experimental measurements.

In random-matrix theory, the Hamiltonian of a physical system is replaced by a matrix whose elements are distributed randomly. The introduction of the concept of random matrix has several motivations. If we assume that the statistical property of eigenstates of a Hamiltonian is determined by the statistical distribution of its matrix elements, then the abstract random matrix is a useful tool to identify the generic property of physical systems which is independent from the peculiarity of each system. Of course, it should be remembered that the relevance of the random matrix to the physical Hamiltonian is, apart from its phenomenological validity, still an open question, which is only partially clarified.[12] A random matrix also helps to reduce the numerical burden, since obtaining good statistics is far easier in random matrices than in physical systems.

We start from the observation that the random scalar number yields the Poisson distribution, and the random rank-2 tensor with real symmetric component; i.e., the Gaussian orthogonal random matrix leads to the Wigner distribution for the nearest-neighbor spacing of its eigenstates sequence. A sparse random matrix with blocks of zero or small numbers in off-diagonal elements can be thought of as the one interpolating these two extremes. By reordering the state indices, one can move these blocks of small elements to the peripheral area of the matrix. Thus we consider a set of *disordered random matrices* in the form

$$H_{ij}=R_{ij}F(|i-j|) \tag{1}$$

as a model to study the transition between Poisson and Gaussian orthogonal ensembles. Here the symmetric matrix R_{ij} has the Gaussian distribution

$$P(R_{ij})=\frac{1}{\sqrt{2\pi(1+\delta_{ij})}}\exp\left[-\frac{R_{ij}^2}{2(1+\delta_{ij})}\right], \tag{2}$$

and the envelope function $F(|i-j|)$ specifies the nature of the distribution. The restriction on F that it depends only on the difference of indices $|i-j|$ can be justified by requiring the ensemble of matrices $\{H_{ij}\}$ to be invariant with respect to the reordering of state indices. We further require that the random matrix H_{ij} has only one global scale; i.e., we exclude such matrices as linear combinations of diagonal random matrices and symmetric random matrices, because such matrices have no well-defined limit when the size of the matrix approaches infinity. Two limits, Poisson and Gaussian orthogonal ensembles, are obtained as special cases with $F(k)=\delta_{k0}$ and $F(k)=1$, respectively. We do not lose generality by assuming $F(k)$ to be a simply decreasing function of k, again because of the possibility of rearranging the state indices. Seligman, Verbaarschot, and Zirnbauer[7] proposed the envelope function of Gaussian form

$$F(k)=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left(-\frac{k^2}{2\sigma^2}\right), \tag{3a}$$

where σ is the parameter controlling the width of the band of "significant" matrix elements around the diagonal line. They were able to fit the eigenvalue distribution of certain physical systems quite well with this rather arbitrary choice. In order to check the sensitivity of eigenstate statistics to the shape of the envelope, we also test the form

$$F(k)=\theta(\eta-k), \tag{3b}$$

which gives a sharp cutoff for nonzero matrix elements within a band along the diagonal line of width $2\eta+1$. For both parametrizations, Eqs. (3a) and (3b), $\sigma=0$ and $\eta=0$ give the Poisson distribution, while $\sigma=\infty$ and $\eta=\infty$ correspond to the Wigner distribution. In reality, for a finite number of the dimension of matrix N, the conditions $\sigma\geq N$ or $\eta\geq N$ are enough to get the Wigner limit.

TABLE I. Six lowest moments of level-spacing distribution in two limiting cases, Poisson and Wigner distributions.

n	Poisson	Wigner
2	2.000	1.273
3	6.000	1.910
4	24.000	3.242
5	120.000	6.079
6	720.000	12.385

Diagonalization of the matrix Eq. (1) gives the sequence of eigenvalues $\{E_i\}$. There are arguments which stress the importance of looking at eigenfunctions rather than the eigenvalues for the signature of quantum chaos.[13] But that subject is beyond the scope of this work. In order to obtain the level-spacing distribution, we have to rescale the spectrum $\{E_i\}$ by unfolding the global level density $\bar{n}(E_i)$. This is achieved[1] by defining the regularized level sequence $\{\varepsilon_i\}$ as

$$\varepsilon_i=\bar{n}(E_i). \tag{4}$$

The average level spacing of the level sequence $\{\varepsilon_i\}$ is uniform and is normalized to be 1. The nearest-level spacing $P_D(x)$, which is the central quantity in this Letter, is defined as

$$P(x)dx=\frac{\text{number of pairs }(\varepsilon_{j+1},\varepsilon_j)\text{ that satisfy } x<(\varepsilon_{j+1}-\varepsilon_j)<x+dx}{\text{total number of pairs sampled}}. \tag{5}$$

It is not practical to look at the full range of the distribution $P(x)$ when one deals with a large number of level sequences. One natural way to characterize the distribution $P(x)$ is by looking at the moments of distribution $M(n)$ which are defined by

$$M(n)=\int_0^\infty dx\, x^n P(x). \tag{6}$$

Usefulness of moments $M(n)$ is seen by calculating the values of Poisson and Gaussian orthogonal ensembles. In Table I, we list from $M(2)$ to $M(6)$. [By definition, $M(0)$ and $M(1)$ are both equal to 1.] We see that the values of $M(n)$ are good indicators of the shape change from Poisson to Wigner distributions.

We are now in the position to state the question of "universal transition" in quantitative terms. Is there any regular pattern in the change of $M(n)$ when the random-matrix parameter σ or η varies? In the absence of the general theory of level spectrum, the only way to answer this question is through numerical experimentation.

We diagonalize the distorted random matrices of both Gaussian- and θ-type envelopes numerically. We took four values 200, 300, 400, and 500 for the dimension of the matrices N. The global level density $\bar{n}(E_i)$ is obtained by fitting the actual levels with polynomials of order up to 8 for each matrix. We collected from 40 to 100 matrices to have roughly about 20000 levels available to calculate moments $M(n)$ of the level-spacing distributions for each value of σ or η. About ten different values between 0 and 50 are chosen for σ and η.

The numerical results are summarized in Figs. 1(a)-1(d). The third, fourth, fifth, and sixth moments are plotted against the second moment in each graph. Open symbols indicate the values of Gaussian form envelope, Eq. (3a), and solid symbols, the θ cutoff, Eq. (3b). Different shaped symbols are for different matrix dimensions.

One sees from the figures that ensembles of both Gaussian- and θ-type envelope, and of different matrix dimensions all result in a common correlated change of different moments. This means that there is a unique path in the transition from Poisson to Wigner distributions for random-matrix ensembles of differing types.

In some literatures,[6,14] the level spacing has been classified and analyzed in terms of the Brody distribution,[11]

$$P_D(x)=(\beta+1)\left[\Gamma\left(\frac{\beta+2}{\beta+1}\right)\right]^{\beta+1}x^{\beta}\times\exp\left\{-\left[\Gamma\left(\frac{\beta+2}{\beta+1}\right)x\right]^{\beta+1}\right\}, \quad (7)$$

where the single parameter β controls the transition from Poisson ($\beta=0$) to Wigner ($\beta=1$) distributions. Since this function was introduced as a purely artificial interpolation formula, it has been somewhat puzzling to see this function being capable of fitting the experimental data as well as theoretically calculated levels. The dashed lines in Figs. 1(a)-1(d) are the predictions for the relation between various moments with the Brody distribution, Eq. (7). Surprisingly, these lines go through the "universal curve" formed by distorted random-matrix ensembles. Although these figures still do not give the physical nor mathematical justification of the Brody distribution, they do give an explanation for its phenomenological validity.

Since the parameters controlling the bandwidth σ and η are attached to the specific (and rather arbitrary) choice of the falloff, they are not suitable for seeing possible universal transition directly. We define the "effective rank" r_c of a matrix by

$$r_c \equiv \log M/\log N\,, \quad (8)$$

where N is the dimension of the matrix, and M is the "effective number" of matrix elements defined as

$$M=\left(\sum_i^N\sum_j^N\langle|H_{ij}|\rangle\right)\Big/\left(\frac{1}{N}\sum_i^N\langle|H_{ii}|\rangle\right), \quad (9)$$

where $\langle\,\rangle$ denotes the ensemble average. For the Poisson ensemble, one gets $r_c=1$, and for the Gaussian orthogonal ensemble, $r_c=2$. For nonzero finite values of σ or η, r_c takes the fractional value between 1 and 2. We emphasize here that Eqs. (8) and (9) are intended not as rigorous definitions but rather as a working hypothesis for a possible scaling parameter. In Fig. 2, we plot the second moment $M(2)$ versus the effective rank r_c. All values indeed seem to form a single narrow band, which becomes almost a single curve toward the Wigner limit,

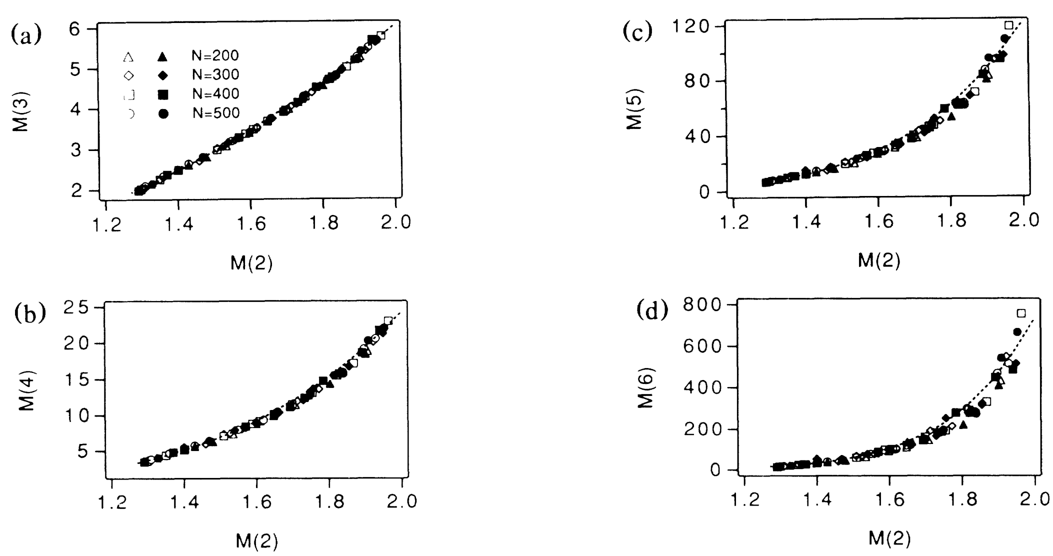

FIG. 1. Relation between various moments of nearest-neighbor level spacings: (a) second vs third moments, (b) second vs fourth, (c) second vs fifth, and (d) second vs sixth. Open symbols show results of the Gaussian envelope, Eq. (3a), and solid symbols, the θ envelope, Eq. (3b). Shapes of the symbols signify the matrix dimensions as shown in the graph. The dashed line is the prediction of the Brody distribution.

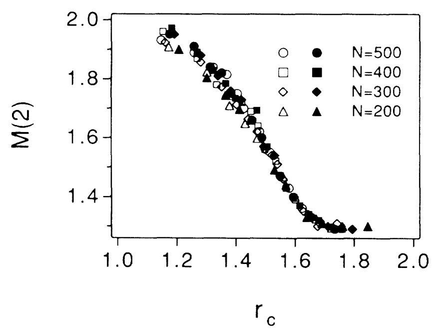

FIG. 2. Second moment of nearest-level-spacing distribution vs effective rank of matrix ensemble defined in Eqs. (8) and (9).

$r_c = 2$. Combining this scaling with the correlations among different moments found in previous graphs, we conclude that there is a wide class of distorted random matrices whose eigenvalue spacing shows the common pattern of transition from Poisson to Wigner distribution. The study of Seligman, Verbaarschot, and Zirnbauer[7] can be interpreted as showing the existence of a physical system (two-dimensional coupled anharmonic oscillator) which actually follows this universal curve. Clearly, more examples should be analyzed before drawing any conclusion about the relevance of our random-matrix result to the physical world.

Although we tested just two examples, the results indicate that it is very unlikely to get a significant deviation from the universal curve by choosing other forms for the envelope function as long as it is smooth and well behaving. We are now checking this point with various forms of $F(k)$. A more fundamental problem is that our prescription for specifying the distorted random matrix is rather loosely formulated, guided only by intuitive arguments without any formal derivation. If more numerical experiments support the universal scaling seen in our examples, the problem of formulating a general theory of distorted random matrices will become a more urgent one.

With all the progress in the study of nonintegrable systems in quantum physics through various numerical experimentations in the last decade, our understanding of quantum signatures of chaotic dynamics is still largely qualitative. It is only quite recently that people started to focus on such quantitative relations as scaling among various indicators of classical and quantum chaos.[10,15] We are in need of identifying the quantities which characterize the degree of order, namely, the order parameter. We hope that our preliminary study here serves as a stepping stone for such an ambitious goal.

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