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Characterizing Loss of Memory in a Dynamical System

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We propose here a new method to characterize the loss of memory with time in a chaotic system from a time series. This is done by introducing time-dependent generalized exponents. The asymptotic behavior can distinguish between chaotic systems which lose memory of the initial conditions completely, those which partially retain the memory, and those (borderline of chaos) which fully retain the memory. We give illustrative examples of the logistic and Hénon maps.

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Sensitivity to initial conditions is one of the most important characteristics of a chaotic attractor.[1] Two close-by trajectories diverge exponentially with time, the rate of divergence being characterized by the Liapunov exponent. Thus a small uncertainty in the initial conditions grows rapidly and after some time it becomes almost impossible to predict the phase-space trajectory. We may say that the system progressively loses the memory of the initial conditions. The time required to lose the memory of the initial conditions depends on the rate of divergence of trajectories and the amount of uncertainty of the initial conditions.

The Liapunov exponent characterizes the exponential divergence of close-by trajectories reasonably well for short times. However, for longer times when the distance between the trajectories approaches the scales of the order of the size of the attractor, the Liapunov exponent is not very useful. In this paper we introduce a new method of analyzing the loss of memory of a chaotic signal which is suitable for all times. The method makes use of the concept of fractal dimension and generalized dimensions[2] and introduces time-dependent generalized exponents. It reduces to an analysis similar to that given by the Liapunov exponent for shorter times. The asymptotic behavior is very interesting and is able to distinguish between different chaotic behaviors.

Consider a time series $\{x_k\}$, $k=1,2,\ldots,$ which specifies the values of an observable x at successive times t_k. We assume that the transients have already died. We first consider a one-dimensional situation. Divide the maximum range of the variable x in N intervals of equal length l. Let p_i be the probability that the variable x lies in the ith interval. We define a joint probability $p_{i,j}(t)$, $i,j=1,2,\ldots,N$, as the probability that the variable x lies in the ith interval at some time t' and in the jth interval at time $t+t'$, i.e., after a time t. The joint probability $p_{i,j}(t)$ will be independent of t' since we assume translational invariance in time.

We now introduce the time-dependent generalized exponents $D_{q,l}(t)$ and $D_q(t)$ for the above cover ($-\infty<q<\infty$) by the following relations:

$$\tau_{q,l}(t)=(q-1)D_{q,l}(t)=\frac{\ln[\sum'_{i,j}p^q_{i,j}(t)]}{\ln l} \tag{1a}$$

and

$$\tau_q(t)=(q-1)D_q(t)=\lim_{l\to 0}\tau_{q,l}(t)\,. \tag{1b}$$

The prime in the summation of Eq. (1a) means that the sum is over only those i and j values for which the probability $p_{i,j}(t)$ is nonzero. $D_{0,l}(t)$, $D_{1,l}(t)$ and $D_{2,l}(t)$ are the time-dependent fractal exponent, information exponent, and correlation exponent, respectively.[2] The exponents that we have defined use the capacity notion of exponents since we use an equal length scale. We also note that $D_q(t)$ defined by Eqs. (1) can be treated as the generalized dimensions in the two-dimensional space defined by the vectors (x_k,x_{k+t}), $k=1,2,\ldots.$

We now consider the situation for finite l. There are two limiting cases. For $t=0$, $p_{i,j}(0)=p_i\delta_{i,j}$. Clearly our time-dependent generalized exponents reduce to the usual time-independent generalized dimensions of the attractor, i.e., $D_{q,l}(0)=D_q$.[3] On the other hand, after a large time interval, if we assume that the system has completely lost the memory of the initial conditions, then we get $p_{i,j}(t)=p_ip_j$. In this case $D_{q,l}(t)=2D_q$. Thus doubling of the generalized exponents will indicate complete loss of memory. We will see that in some situations this doubling does not occur, i.e., the memory is never completely lost even when we have exponentially diverging trajectories as indicated by a positive Liapunov exponent.

We note that our condition of complete loss of memory corresponds to the mixing property which states that $\lim_{t\to\infty}\Pr(\phi^{-t}B\cap A)=\Pr(B)\Pr(A)$ for all sets A and B with ϕ^t a dynamical map.[4]

It is possible to relate the time-dependent fractal exponent $D_{0,l}(t)$, the Liapunov exponent λ, and the length scale l. In t time steps an interval of width l will be mapped into length $R=le^{\lambda t}$. If for times larger than some time $\bar{t}$, R becomes of the order of the size of the attractor, we can roughly say that the memory of the initial conditions is completely lost; i.e., given an uncertainty of l in the starting value of the variable x, the value of x after time $\bar{t}$ may lie anywhere in the attractor and is thus completely unpredictable. The size of the attractor when measured with the scale l is $l^{-D_0}l$. Thus we get the relation[5]

$$\bar{t}=-D_0(\ln l)/\lambda\,. \tag{2}$$

For time $t<\bar{t}$, R is less than the size of the attractor. The number of lengths, that R covers is R/l. If we start from N initial lengths, they are mapped into NR/l lengths. Starting with the entire attractor, i.e., $N=l^{-D_0}$, we get

$$D_{0,l}(t)=D_0-\lambda t/\ln l\,. \tag{3}$$

We now illustrate the use of our formalism by considering the example of the logistic map given by $x_{n+1}=\mu x_n(1-x_n)$. We have analyzed the chaotic data for three different values of μ.

(a) $\mu=4.0$.—For this value of μ the logistic map shows a fully developed chaos. The values of $\tau_{q,l}(t)$ as a function of q for different times are shown in Fig. 1 for $l=0.01$. We see that around $t=9$, $\tau_{q,l}(t)$ values reach about twice the values at $t=0$. For $q>0$, $\tau_{q,l}(t)$ increase steadily as a function of time. However, for $q<0$, $\tau_{q,l}(t)$ values decrease rapidly and then increase. These values correspond to low probabilities. In this region ($q<0$) numerical errors are larger and it is difficult to interpret the results. The fractal exponent $D_{0,l}(t)$ as a function of t is shown in Fig. 2 for different values of l. There is a sharp rise and subsequent flattening as we approach $D_{0,l}(t)\simeq 2$. The time required to reach the value 2 obviously depends on l. The rise from $t=0$ to $t=1$ is an artifact of our choosing equal length scales and not the natural length scales of the system. The region of steady rise of $D_{0,l}(t)$ can be used to obtain the Liapunov exponent by using Eq. (3). We get the average $\lambda\simeq 0.60$,

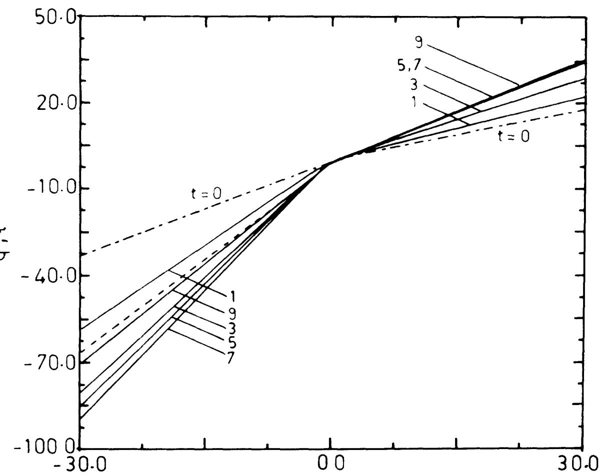

FIG. 1. $\tau_{q,l}(t)$ vs q plots for different times for the logistic map with $\mu=4.0$ and $l=0.01$. The two dashed curves correspond to the $\tau_{q,l}(t)$ values for $t=0$ and values obtained by doubling its values (Ref. 6). The solid curves correspond to $\tau_{q,l}(t)$ values for different times. The number of points of the time series chosen in this and all the other calculations is such that each box has at least forty points on the average.

while the exact analytical value is $\lambda=\ln 2=0.69$. In Fig. 2 we also show the behavior of $D_{0,l}(t)$ for $l=0.0033$ expected from Eq. (3) with $\lambda=\ln 2$ (dashed line). We see that the numerical values show a systematic deviation. They are larger than those given by Eq. (3) for small t and smaller for large t. Hence the estimated value of the Liapunov exponent is smaller than the actual value and serves as a lower bound. We also find that the estimate

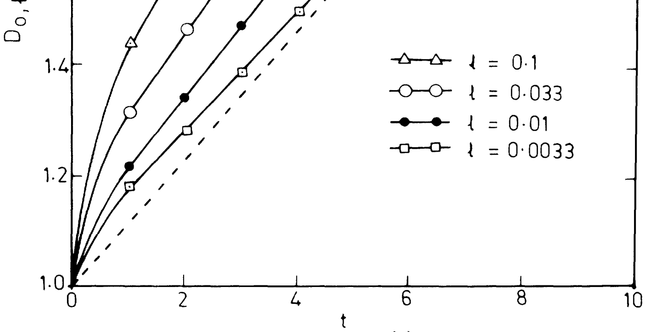

FIG. 2. The fractal exponent $D_{0,l}(t)$ as a function of t for four different values of l for the logistic map with $\mu=4.0$. The dashed line shows the behavior of $D_{0,l}(t)$ for $l=0.0033$ and $\lambda=\ln 2$ according to Eq. (3).

of λ improves as l decreases.

(*b*) $\mu = 3.5699\ldots$—This value of μ gives the period-doubling attractor. The Liapunov exponent is zero. Thus we expect that fractal exponent $D_{0,l}(t)$ should remain invariant in time. We find that this is indeed the case. [See the triangles in Fig. 3. There are some small oscillations which may be due to not using natural length scales and the $D_{0,l}(t)$ values are larger than the fractal dimension for the attractor due to the finite value of l.] This indicates that there is no loss of memory and the future is completely predictable.[7]

(*c*) $\mu = 3.59687$.—This value of μ gives a two-band attractor. The points represented by circles in Fig. 3 show $D_{0,l}(t)$ as a function of t for $l = 0.01$. The behavior for small t is similar to the $\mu = 4$ case. However, the asymptotic behavior is quite different. First, the asymptotic value of the fractal exponent is clearly less than twice its value at $t = 0$. This shows that the system never loses memory of the initial conditions completely. This result may be because the chaos is not fully developed. The variable x_n keeps on alternating between the two bands and we always know the band in which it lies at any time once the initial band is known. Second, $D_{0,l}(t)$ alternates between the two values asymptotically. This is because the two bands have different widths and we have used the capacity notion of dimensions. For small t a straight-line fit to the data in Fig. 3 yields $\lambda = 0.13$ in comparison to the actual value 0.17.

Let us now consider the limit $l \to 0$. From Eq. (3) we see that the slope of $D_{0,l}(t)$ vs t tends to 0 as $l \to 0$ and hence for any finite t, $D_{0,l}(t)$ tends to D_0 and not $2D_0$ as $l \to 0$. The slopes of actual curves of $D_{0,l}(t)$ in Fig. 2 are also decreasing as $l \to 0$ and seem to support this conclusion. Based on this let us conjecture that $\lim_{l \to 0} D_{q,l}(t) = D_q$. This implies that there is no loss of memory if the initial conditions are specified with infinite precision. This is natural since we have deterministic chaos. Thus the above conjecture allows us to conclude that the loss of memory of the initial conditions is a property of coarse graining.[8] We note that the two limits $l \to 0$ and $t \to \infty$ are not interchangeable. Taking the $l \to 0$ limit first and the $t \to \infty$ limit afterwards gives us the generalized dimensions D_q. On the other hand, taking the $t \to \infty$ limit first we get the asymptotic behavior discussed for the finite-l case above.

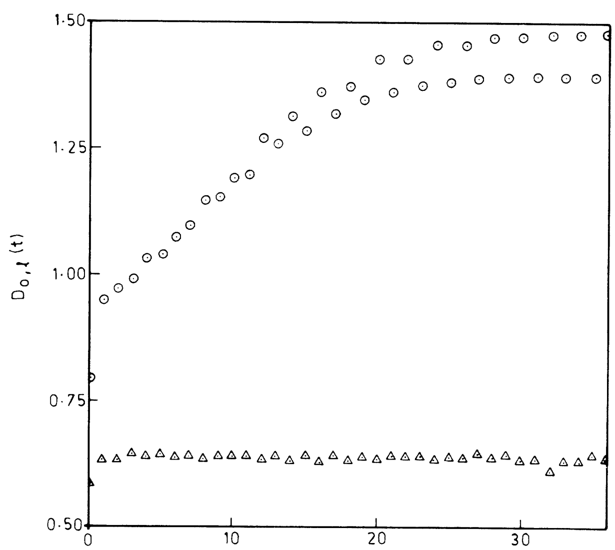

FIG. 3. The fractal exponent $D_{0,l}(t)$ vs t for the two-band case of logistic map for $l = 0.01$ (○). △'s are similar points for the period-doubling attractor.

We now consider higher-dimensional systems. For a d-dimensional system, Eqs. (1) can be easily generalized by letting the indices i and j represent d-dimensional boxes. If we have a time series in only one variable, we can use the method of time delays to construct the state vectors $\mathbf{x}_k = (x_k, x_{k+1}, \ldots, x_{k+d-1})$, $k = 1, 2, \ldots$, where d is the embedding dimension.[9] The index i for the box is replaced by the string $(i_1, i_2, \ldots, i_d)$, where i_m is the index for the length intervals in the mth direction of the embedding space. The time-dependent generalized exponents are now given by Eqs. (1) with the summation indices i and j now representing d-dimensional boxes, $(i_1, i_2, \ldots, i_d)$ and $(j_1, j_2, \ldots, j_d)$, respectively. Again, for $t = 0$ we get the usual generalized dimensions of the attractor and for large times, if memory is completely lost, we get doubling of the exponents.

However, this procedure is cumbersome for actual calculations of $D_q(t)$ since it requires working in a space of dimension $2d$. A variant of time-dependent generalized exponents for $d > 1$ which gives the same information can be introduced by defining the modified time-dependent generalized exponents $D_{q,l}(t)$ as

$$D_{q,l}(t) = \frac{1}{q-1} \frac{\ln\{\sum_{(i_1, \ldots, i_d), j}[p_{(i_1, \ldots, i_d), j}(t)]^q\}}{\ln l}, \qquad (4)$$

where $p_{(i_1, \ldots, i_d), j}(t)$ is the probability that the state vector $\mathbf{x}$ lies in the box $(i_1, \ldots, i_d)$ at some time t' and the variable x lies in the length interval j after a time $t + d - 1$, i.e., at time $t + t' + d - 1$. This definition of $D_{q,l}(t)$ requires calculations in a space of $d + 1$ dimensions only. For $t = 0$, we again get the usual generalized dimensions of the attractor, i.e., D_q. In the other limit for large time if we assume that the variable x completely loses memory of the initial-state vector, then

$$p_{(i_1, \ldots, i_d), j}(t) = p_{(i_1, \ldots, i_d)} p_j .$$

For a chaotic attractor with $D_0 > 1$, the projection along any direction is expected to be continuous. Thus, in this case $D_{q,l}(t) = D_{q,l} + 1$.

We now consider the example of a two-dimensional map, namely, the Hénon map, $(x_{n+1}, y_{n+1}) = (y_n + 1 - a x_n^2, b x_n)$, with $a = 1.4$ and $b = 0.3$. Let us consider the time series in only one variable, say, x. We next construct the state vectors $\mathbf{x}_k = (x_k, x_{k+1})$. Using this state vector the time-dependent generalized exponents can be calculated. We have carried out this calculation by both

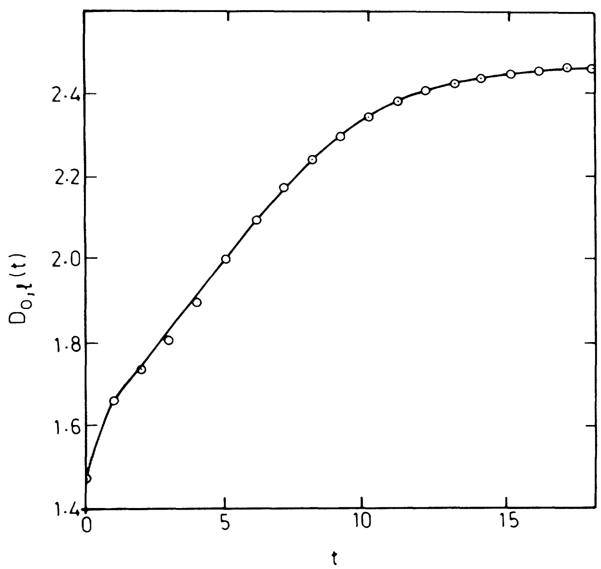

FIG. 4. $D_{0,l}(t)$ as a function of t for $l=0.05$ for the Hénon map.

the methods described above. First, we use Eq. (1a), i.e., use two-dimensional boxes for the indices i and j. We find that the fractal exponent $D_{0,l}(t)$ doubles asymptotically. This shows that loss of memory of the initial conditions is complete in the Hénon map. The value of the Liapunov exponent obtained from the slope of $D_{0,l}(t)$ vs t curve is 0.32 [see Eq. (3)]. The known largest value of λ for the Hénon map is 0.418.[10]

Next we use Eq. (4) and obtain the modified exponents $D_{q,l}(t)$. Figure 4 shows $D_{0,l}(t)$ as a function of t for $l=0.05$. The increase of 1 in fractal dimension confirms the fact that there is complete loss of memory. The value of the Liapunov exponent obtained by using Eq. (3) is 0.33.

In this paper we have presented a new method of analyzing the time evolution of a chaotic signal. In particular, we know how the loss of memory of the initial conditions takes place at each time step. The change in the time-dependent fractal exponent is a measure of this loss of memory. Complete loss of memory is represented by doubling of the time-dependent generalized exponents. Our method is able to distinguish between chaotic signals which lose memory completely and those which retain partial memory of the initial conditions. Thus we expect our method to be useful in knowing the kind of chaotic attractor that one has. We do not know of any other simple method which can do this. In addition, we also get a rough estimate (actually a lower bound) of the Liapunov exponent. We also note that the loss of memory appears to be a property of coarse graining.

Our analysis has also relevance for predicting a chaotic time series. Farmer and Sidorowich[11] find that the normalized error of prediction approaches 1 for large prediction times in some systems while it remains less than 1 in other cases. These two situations will correspond to an asymptotic value of $D_{q,l}(t)$ which is twice the original value and an asymptotic value which is less than twice the original value, respectively.

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