Simple Unified Form for the Major No-Hidden-Variables Theorems

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(Received 9 October 1990)

Two examples are given that substantially simplify the no-hidden-variables theorem of Kochen and Specker, greatly reducing the number of observables considered and requiring no intricate geometric argument. While one of the examples also obeys a more powerful version of Bell's theorem, the other does not. The examples provide a new perspective on both of these fundamental theorems and on the relation between them.

PACS numbers: 03.65.Bz

The Kochen-Specker (KS) theorem[1] demonstrates that it is, in general, impossible to ascribe to an individual quantum system a definite value for each of a set of observables not all of which necessarily commute. Of course elementary quantum metaphysics insists that we cannot assign definite values to noncommuting observables; the point of the KS theorem is to extract this directly from the quantum-mechanical formalism, rather than merely appealing to precepts enunciated by the founders. If such an assignment of values turned out to be possible in spite of those precepts,[2] then uncertainty relations for the results of measuring noncommuting observables could be viewed as a manifestation of the statistical scatter of these definite values in many different individual realizations of the identical quantum state. The state vector alone would not provide complete information about a system, and the additional values in particular realizations of the same quantum state could be regarded as "hidden variables."

A similar conclusion against hidden variables is reached by Bell's Theorem,[3] but in a rather different way. The violation of quantum dogma contemplated by Bell is weaker than that tested by Kochen and Specker, noncommuting observables only being provided with simultaneous values when required to have them by the simple locality condition of Einstein, Podolsky, and Rosen (EPR).[4] On the other hand, Bell's refutation has a strongly statistical character that the argument of Kochen and Specker does not.

I describe below a simple system for which one can prove both a KS and a Bell-EPR theorem. The KS theorem is substantially simpler than the original argument of Kochen and Specker; the Bell-EPR theorem eliminates the statistical aspect of Bell's original argument; and the applicability of both theorems to a single system clarifies their relationship, clearly revealing the Bell-EPR result to be the stronger of the two.

The discussion that follows is inspired by a new version of Bell's Theorem due to Greenberger, Horne, and Zeilinger (GHZ),[5-9] by the observation of Stairs[10] that GHZ can also be made the basis for a KS theorem, and by a further simplification of the GHZ version of the KS theorem due to Peres.[11] I first describe the assumptions tested by the KS theorem and the argument of Peres which disposes of them in an extraordinarily simple way. I then give a refinement of Peres's argument, and show how an independent but analogous argument of Stairs extracts a very simple KS theorem for a GHZ example. Finally, I show that while the GHZ example obeys a Bell-EPR theorem as well as a KS theorem, the example of Peres does not work as a Bell-EPR theorem, for reasons that illuminate the structure of the Bell-EPR argument.

Before we start entertaining heretical notions about preexisting values for observables, let us note, as a piece of entirely orthodox quantum mechanics, that if some functional relation

$$f(A,B,C,\dots)=0 \qquad (1)$$

holds as an operator identity among the observables of a mutually *commuting* set, then since the results of the simultaneous measurements of $A,B,C\dots$ will be one of the sets $a,b,c\dots$ of simultaneous eigenvalues of $A,B,C,\dots,$ the results of those measurements must also satisfy

$$f(a,b,c,\dots)=0\,, \qquad (2)$$

whatever the state of the system prior to the measurement.

With this in mind, let us call the values we would like to try to ascribe to the observables A, B, $C,\dots$ for an individual system $v(A)$, $v(B)$, $v(C),\dots.$ Since these are to be the values revealed by a subsequent measurement, the value associated with an observable must be one of its eigenvalues. Since any commuting subset of the full set of observables can be measured simultaneously, if the values are to agree with the predictions of quantum mechanics they must be constrained by the condition that any relation $f(A,B,C,\dots)=0$ holding as an identity among operators in a commuting subset must also hold for their values:

$$f(v(A),v(B),v(C),\dots)=0\,. \qquad (3)$$

The KS theorem proves that such an attempt to assign values cannot, in general, succeed.[12] Their counterexample consists of a three-dimensional state space, which can be viewed as describing the spin states of a spin-1 particle. The observables to be assigned values are squares of spin components along different directions. It is an elementary property of the spin-1 angular momentum operator that the square of its component along any direction has eigenvalues 0 or 1, that the squares of its components along two orthogonal directions commute, and that the sum of the squares of its components along three mutually orthogonal directions is just the c-number 2. Kochen and Specker originally produced a set of 117 observables, associated with the squares of the components of the angular momentum operator along 117 different directions.[13] They demonstrate with a somewhat intricate geometrical argument that there is no way to assign the values 0 or 1 to all these observables, consistent with the requirement that $v(A)+v(B)+v(C)=2$ for any set of observables A, B, and C associated with three mutually orthogonal directions. To the well trained quantum mechanician it must surely seem shocking that the direct refutation of so heretical an attempt should require so elaborate a counterexample, but that is where things have stood for almost 25 years. We do not pursue the intricacies of the KS demonstration here, but turn to some new and very much simpler counterexamples.

Peres has found a spectacular simplification in the four-dimensional space of two spin-$\frac{1}{2}$ particles. Consider an attempt to assign values to the six operators $\sigma_x^1\sigma_y^2$, $\sigma_y^1\sigma_x^2$, σ_x^1, σ_x^2, σ_y^1, σ_y^2. This attempt must fail if the particles are in the singlet state, for to agree with the predictions of quantum mechanics in that state, the values must satisfy $v(\sigma_x^2)=-v(\sigma_x^1)$ and $v(\sigma_y^2)=-v(\sigma_y^1)$. But we must also have $v(\sigma_x^1\sigma_y^2)=v(\sigma_x^1)v(\sigma_y^2)$ and $v(\sigma_y^1\sigma_x^2)=v(\sigma_y^1)v(\sigma_x^2)$, since each of these identities merely asserts that the values of mutually commuting operators satisfy the same identity as the operators. It follows that $v(\sigma_x^1\sigma_y^2)v(\sigma_y^1\sigma_x^2)=1$. But $\sigma_x^1\sigma_y^2$ and $\sigma_y^1\sigma_x^2$ commute with each other and with their product, which is $\sigma_z^1\sigma_z^2$; this requires $v(\sigma_x^1\sigma_y^2)v(\sigma_y^1\sigma_x^2)$ to be equal to $v(\sigma_z^1\sigma_z^2)$, which is -1 in the singlet state. Therefore values cannot be assigned.

While Peres's argument is enormously simpler than that of Kochen and Specker, unlike theirs, his counterexample relies on the properties of a particular state. While this in no way diminishes its power as a counterexample,[14] one might wonder whether it is the state independence of the original KS argument that requires its much greater complexity. We therefore note next that Peres's argument can easily be recast in a form that strictly parallels the form of the KS argument,[15] making no reference to properties special to a particular state. We need only add the three operators $\sigma_x^1\sigma_x^2$, $\sigma_y^1\sigma_y^2$, $\sigma_z^1\sigma_z^2$ to Peres's set of six, and note that six elementary identities within mutually commuting subsets of the nine operators lead, independent of the state of the system, to the following six constraints on the values assigned to those mutually commuting operators:[16]

$$\begin{aligned} &v(\sigma_x^1\sigma_x^2)v(\sigma_x^1)v(\sigma_x^2)=1\,,\\ &v(\sigma_y^1\sigma_y^2)v(\sigma_y^1)v(\sigma_y^2)=1\,,\\ &v(\sigma_x^1\sigma_y^2)v(\sigma_x^1)v(\sigma_y^2)=1\,,\\ &v(\sigma_y^1\sigma_x^2)v(\sigma_y^1)v(\sigma_x^2)=1\,,\\ &v(\sigma_x^1\sigma_y^2)v(\sigma_y^1\sigma_x^2)v(\sigma_z^1\sigma_z^2)=1\,,\\ &v(\sigma_x^1\sigma_x^2)v(\sigma_y^1\sigma_y^2)v(\sigma_z^1\sigma_z^2)=-1\,. \end{aligned} \tag{4}$$

Since the eigenvalues of all nine operators are 1 or -1, since each value $v(A)$ must be an eigenvalue of the operator A, and since each value appears twice on the left of Eqs. (4), the product of all the left sides is 1. Since the product of the right sides is -1 the assignment of values is impossible. We have thus reduced the KS impossibility proof from a consideration of dozens of operators (in three dimensions) and some extensive geometrical analysis, to a consideration of nine operators (in four dimensions) and some trivial arithmetic.

Although this nine-operator four-dimensional simplification of KS is about as elementary as one could hope to achieve, we shall see below that it works only as a KS argument, and not as a Bell-EPR argument. In contrast, the three-particle version of the Bell-EPR example of Greenberger, Horne, and Zeilinger can also be cast in a form that strictly parallels the KS argument with ten operators in eight dimensions, while retaining the analytical simplicity of the Peres example. It thereby provides a new and instructive insight into the relation between the argument of Kochen and Specker, and Bell's analysis of the EPR experiment.[17]

We work in the eight-dimensional space of three spins $\frac{1}{2}$, and attempt to assign values to the ten operators σ_x^1, σ_y^1, σ_x^2, σ_y^2, σ_x^3, σ_y^3, $\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, $\sigma_y^1\sigma_y^2\sigma_x^3$, and $\sigma_x^1\sigma_x^2\sigma_x^3$. Identities among operators in mutually commuting subsets now lead to five constraints on these values:[16]

$$\begin{aligned} &v(\sigma_x^1\sigma_y^2\sigma_y^3)v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^3)=1\,,\\ &v(\sigma_y^1\sigma_x^2\sigma_y^3)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_y^3)=1\,,\\ &v(\sigma_y^1\sigma_y^2\sigma_x^3)v(\sigma_y^1)v(\sigma_y^2)v(\sigma_x^3)=1\,,\\ &v(\sigma_x^1\sigma_x^2\sigma_x^3)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_x^3)=1\,,\\ &v(\sigma_x^1\sigma_x^2\sigma_x^3)v(\sigma_x^1\sigma_y^2\sigma_y^3)v(\sigma_y^1\sigma_x^2\sigma_y^3)v(\sigma_y^1\sigma_y^2\sigma_x^3)=-1. \end{aligned} \tag{5}$$

Once again, since the eigenvalues of all ten operators are 1 or -1 and since each value appears twice on the left of Eqs. (5), the products of all the left sides must be 1 and the assignment of values is impossible.

The virtue of this example is that it not only works as an extremely simple KS argument, but can also be used to give a Bell-EPR argument.[18] To use GHZ in the Bell-EPR context we must consider a system in a particular one of the simultaneous eigenstates of the three commuting operators $\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, and $\sigma_y^1\sigma_y^2\sigma_x^3$ —say the state Φ in which all three have eigenvalue 1.[19] It follows that Φ is also an eigenstate of $\sigma_x^1\sigma_x^2\sigma_x^3 = -(\sigma_x^1\sigma_y^2\sigma_y^3)(\sigma_y^1\sigma_x^2\sigma_y^3)(\sigma_y^1\sigma_y^2\sigma_x^3)$ with eigenvalue -1. We now note that if three mutually well separated particles have their spins in this state, then we can learn in advance the result m_x of measuring the x component of the spin of any one of them by far away measurements of the y components of the other two, since the product of all three measurements in the state Φ must be unity. For the same reason we can learn in advance the result m_y of measuring the y component of any one of them by far away measurements of the x component of a second and the y component of a third. If (like Einstein) one is afflicted with a strong antipathy toward nonlocal influences, then one is impelled to conclude that the results of measuring either component of any of the three particles must have already been specified prior to any of the measurements—i.e., that any particular system in the state Φ must be characterized by numbers m_x^1, m_y^1, m_x^2, m_y^2, m_x^3, m_y^3 which specify the results of whichever of the four different sets (xyy, yxy, yyx, or xxx) of three single-particle spin measurements one might choose to make on the three far apart particles. Because, however, Φ is an eigenstate of $\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, $\sigma_y^1\sigma_y^2\sigma_x^3$, and $\sigma_x^1\sigma_x^2\sigma_x^3$ with respective eigenvalues 1, 1, 1, and -1, the products of the four trios of 1's or -1's must satisfy the relations

$$\begin{aligned} &m_x^1m_y^2m_y^3=1, \quad m_y^1m_x^2m_y^3=1\,, \\ &m_y^1m_y^2m_x^3=1, \quad m_x^1m_x^2m_x^3=-1\,, \end{aligned} \tag{6}$$

which, once again, are mutually inconsistent, the product of the four left sides being necessarily +1.

Evidently GHZ is a stronger argument in its Bell-EPR form than in its KS form. The KS form refutes the attempt to assign values to arbitrary observables; the Bell-EPR form refutes the attempt to assign values only to those observables required to have them by Einstein locality. Of course the Bell-EPR form of the argument, unlike the KS form, depends critically on the properties of certain particular states, but the fact that the stronger KS assumptions lead to a stronger refutation (that works in an arbitrary state) is immaterial: A refutation is a refutation.

A comparison between the original forms of Bell-EPR and KS is not as transparent because Bell's Theorem, unlike KS or the theorem of Bell in Ref. 1, does not demonstrate a direct inconsistency between the values required to exist by Einstein locality, appealing instead to an inconsistency in the statistical distribution of results those values imply for many runs of an additional experiment. I suspect it is for this reason (and also because of its independence of any particular state) that the KS theorem has continued to be of interest right along with Bell's Theorem. The strengthening of Bell's example by GHZ and the demonstration that GHZ also works as a KS theorem should liberate future generations of students of the foundations of quantum mechanics from having to cope with the geometrical intricacies of the original KS argument.[20]

Note, finally, how things go wrong when we try to cast the Peres version of Kochen and Specker into the form of a Bell-EPR argument. If the system is in the singlet state, Einstein locality requires us to preassign values m_x^1, m_y^1, m_x^2, and m_y^2 to all the results of measuring either the x or y component of the spin of either particle, and these values are constrained by the relations $m_x^2=-m_x^1$ and $m_y^2=-m_y^1$. Einstein locality also requires that the results of simultaneously measuring σ_x^1 and σ_y^2 should be given by m_x^1 and m_y^2, and the results of simultaneously measuring σ_y^1 and σ_x^2, by m_y^1 and m_x^2. If we were allowed to assign simultaneous values to the two commuting observables $\sigma_x^1\sigma_y^2$ and $\sigma_y^1\sigma_x^2$ the chain would be complete. On the one hand those values would have to be $m_x^1m_y^2$ and $m_y^1m_x^2$ whose product is 1. But on the other hand the product $(\sigma_x^1\sigma_y^2)(\sigma_y^1\sigma_x^2)$ is $\sigma_z^1\sigma_z^2$ of which the singlet state is an eigenstate with eigenvalue -1, so the product of the results of simultaneously measuring the two observables in the singlet state is necessarily -1: a contradiction. But although it is legitimate to assign values to both of these observables in making a KS argument, there is no reason to do so on the basis of Einstein locality alone. Any procedure that simultaneously measures the commuting observables $\sigma_x^1\sigma_y^2$ and $\sigma_y^1\sigma_x^2$ for two well separated particles must be highly nonlocal, so there are no intuitive grounds for inferring from the perfect correlations between the values such a measurement produces that Einstein locality requires each observable independently to have a definite value prior to the measurement.

The GHZ example does not encounter this obstacle because the four nonlocal observables it considers ($\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, $\sigma_y^1\sigma_y^2\sigma_x^3$, and $\sigma_x^1\sigma_x^2\sigma_x^3$) are mutually commuting. They can therefore all be assigned definite values by the appropriate choice of state vector, which circumvents the inability to assign them values by an appeal to Einstein locality. The four nonlocal observables in my version of the Peres example ($\sigma_x^1\sigma_x^2$, $\sigma_y^1\sigma_y^2$, $\sigma_x^1\sigma_y^2$, and $\sigma_y^1\sigma_x^2$), however, are not mutually commuting, so no choice of state vector can supply the missing link in the argument. The crucial role played by the specification of a state vector in the Bell-EPR argument is thus laid bare.

I am indebted to Allen Stairs and Asher Peres for sending me prepublication copies of their arguments, and to Peres for an enlightening electronic-mail correspondence in response to my initial reaction that his beautiful example was wrong. I am grateful to Abner Shimony for teaching me more of the history of the subject, and insisting that I characterize accurately the full power of the Gleason-Bell-KS results. But like others who puzzle

over the meaning of quantum mechanics, I am obligated most of all to John Bell, whose insight, wit, and moral fervor we shall all sorely miss.

[1]S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967). A similar result was independently derived by J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966), but I refrain from using the term "Bell-KS theorem" to avoid confusion with Bell's much more famous theorem, which also plays an important role in what follows. Both the KS and the Bell arguments are versions of a result first derived by A. M. Gleason, J. Math. Mech. **6**, 885 (1957).

[2]It *is* possible for a two-dimensional quantum system—i.e., a single spin $\frac{1}{2}$. See, for example, Bell (Ref. 1).

[3]J. S. Bell, Physics **1**, 195 (1964). By "Bell's Theorem" (or "Bell-EPR") I shall always mean this celebrated result, rather than the theorem of Bell cited in Ref. 1 (which was proved before Bell's Theorem, despite the dates of publication: see footnote 19 of the Bell citation in Ref. 1).

[4]A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935). I follow the common practice of referring to this as "Einstein locality."

[5]D. M. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe,* edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.

[6]R. K. Clifton, M. L. G. Redhead, and J. N. Butterfield (to be published).

[7]N. D. Mermin, Am. J. Phys. **58**, 731 (1990).

[8]N. D. Mermin, Phys. Today **43**(6), 9 (1990).

[9]D. M. Greenberger, M. Horne, A. Zeilinger, and A. Shimony, Am. J. Phys. **58**, 1131 (1990).

[10]A. Stairs (private communication).

[11]A. Peres, Phys. Lett. A **151**, 107 (1990).

[12]Implicit in the attempt, and what brings it to grief, is the assumption that $v(A)$ is independent of the particular set of simultaneously measured mutually commuting observables to which A belongs.

[13]The number necessary for the proof is reduced to 109 by A. Peres and A. Ron, in *Microphysical Reality and Quantum Formalism,* edited by A. van der Merwe, F. Selleri, and G. Tarozzi (Kluwer-Academic, Dordrecht, 1988), Vol. 2, p. 115. S. Kochen tells me that he and J. Conway have very recently reduced the number to 33.

[14]On the other hand Peres's argument is substantially weaker than the KS theorem (or the arguments of the KS type that I give below) in the following sense: Suppose, given an individual system, that you try to make a list of values for all of its observables. The KS theorem shows that among the sets of observables you can simultaneously measure, there is at least one set which, if measured on that individual system, must give values in disagreement with those on your list. Peres's theorem establishes this only for those individual systems taken from an ensemble of systems prepared in the singlet state, leaving open the question of whether there might not be other ways to prepare an individual system for which the trick could be accomplished. The KS theorem says that you can never do it, no matter how the individual system has been prepared.

[15]It is also possible, though more awkward, to phrase everything in terms of projection operators.

[16]The original operator identities can be seen, of course, by simply deleting the v's.

[17]What follows is very close to the Bell-EPR version of GHZ I give in Refs. 7 and 8, whose relevance to a KS argument was brought to my attention by A. Stairs.

[18]The Bell-EPR use of the GHZ example is described briefly below, but in pedagogical detail in Ref. 7.

[19]Greenberger, Horne, and Zeilinger invented their example to give a Bell-EPR argument. A clue that it might also yield a KS argument making no reference to a particular state is that the Bell-EPR argument can be made regardless of which of the eight simultaneous eigenstates the system is in.

[20]The original KS argument is, of course, required if one wishes to make the point in a three-dimensional state space. For four or more dimensions the point can be made more simply by constructing my generalization of Peres's argument in any four-dimensional subspace, and for eight or more dimensions, one can construct a GHZ argument in any eight-dimensional subspace.