Semiclassical Theory of Localized Many-Anyon States

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We study the bound-state spectrum of many particles obeying fractional statistics (anyons) in a two-dimensional system in the presence of an external potential and a large magnetic field B. As a function of B, there occur periodic recurrences of quantized states at fixed energy. For fixed particle number, the period is determined by the fractional charge, while at fixed chemical potential, the fractional charge is masked by the fractional statistics. We discuss the relevance of these results to recent experiments in quantum Hall devices, and we propose a simple extension of these experiments which could measure the fractional statistics.

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Because the configuration space of a two-dimensional gas of identical particles is multiply connected, it is possible to define[1,2] a consistent quantum theory in which the particles ("anyons") obey fractional statistics intermediate between Bose and Fermi statistics. Such particles are characterized by the statistical angle θ^* which is the phase change of the wave function when one particle traverses a path which encloses another in the clockwise direction; $\theta^*=0(\text{mod}4\pi)$ corresponds to Bose and $\theta^*=2\pi(\text{mod}4\pi)$ to Fermi statistics. In particular, in the fractional quantum Hall state, the quasiparticle excitations are believed to have fractional charge[3] $e^*=e/m$, and corresponding fractional statistics[4,5] $\theta^*=2\pi/m$; indeed gauge invariance implies that a connection of this sort between charge and statistics must occur[6] whenever fractionally charged quasiparticles occur.

Fractional statistics greatly complicate the solution of the many-body problem, and even the several-body problem, due to the long-range nature of the statistical interaction. In particular, although for noninteracting bosons or fermions the many-body wave function can be constructed as a product of solutions of the one-body problem, this is not the case for particles with intermediate statistics. Thus, even the problem of multiple, noninteracting anyons is nontrivial.

Semiclassical methods provide a simple and physically transparent method for understanding bound states of complicated systems. In two dimensions in the limit of a large magnetic field B, when the cyclotron energy, $\hbar\omega_c=\hbar eB/Mc$ (M is the particle's effective mass), is large compared to the potential energy, all low-energy states are restricted to lie in the lowest Landau level. The remaining dynamics of the particles involves only the guiding-center motion, and hence is independent of M. The quantum theory of the guiding-center dynamics can be expressed in terms of a coherent-state path integral;[7] the semiclassical theory is then the first term in an asymptotic expansion in powers of $(l/R)^2$, where l is the Landau length, $l^2=\phi_0/2\pi B$, $\phi_0=hc/e$ is the flux quantum, and R is the length scale characteristic of changes in the potential energy. Extensive comparisons have been made[7(b)] between semiclassical expressions and exact numerical solutions of model problems in the high-field limit, and the semiclassical theory has been found to be remarkably accurate.

In particular, we have in mind obtaining a more detailed understanding of the recent experiments of Simmons *et al.*[8] on narrow quantum Hall devices which have yielded the first direct experimental evidence of the existence of well-defined quasiparticle excitations with fractional charge. In these experiments, periodic oscillations of the longitudinal resistance ρ_{xx} were observed as a function of magnetic field near the edges of various quantum Hall plateaus. For the integer plateaus, with Hall conductance $\sigma_{xy}=(e^2/h)\nu$, where $\nu=1,2,3,4$, the period of the oscillations in ρ_{xx} was found to be roughly independent of ν, while for $\nu=\frac{1}{3}$, the period was 3 times as large as for integer ν. A simple explanation for this observation is suggested[9] by considering the semiclassical bound-state spectrum of a single particle of charge e^* in a smoothly varying external potential. If there exists a bound state with energy E_F at some value of the magnetic field B, then when the magnetic field is $B+\Delta B$ there again exists a state with the same energy provided

$$\Delta B=\phi^*/2\pi A(E_F)\ , \tag{1}$$

where $\phi^*=hc/e^*$ is the effective flux quantum, and $A(E)$ is the area enclosed by the equal-potential contour with energy E. Thus, if $A(E_F)$ is approximately independent of magnetic field, and if the period of the resistance oscillations in ρ_{xx} is determined by this resonance condition, then the resistance oscillations are sensitive only to the charge of the quasiparticle which is $e^*=e$ for any integer plateau and $e^*=e/3$ for the $\frac{1}{3}$ state.[10]

There is an apparent paradox implicit in Eq. (1): The quantization condition is superficially reminiscent of the behavior of a system with an Aharonov-Bohm geometry, in which magnetic flux is passed through a region of the system which is inaccessible to the particles. The fact that, at a microscopic level, the system is composed of electrons with charge e, implies that, in a true

Aharonov-Bohm experiment, all properties of the system[11] must be strictly periodic functions of the flux through the hole with period ϕ_0. Since for $e^* = \frac{1}{3}$ we have $\phi_0 < \phi^*$, this periodicity would appear to be inconsistent with the observation of a ϕ^* periodicity of the resistance. We will find that there is no paradox; the experiment does not measure the Aharonov-Bohm effect. In particular, Eq. (1) was derived under the condition that there is only one quasiparticle in the system.[12] Thus, the entire region enclosed by the quasiparticle orbit must be covered by condensate which is affected by the magnetic field. We will show that as a consequence of the fractional statistics of the quasiparticles the expected ϕ_0 periodicity of observable quantities for Aharonov-Bohm conditions emerges from the many-particle generalization of Eq. (1). The same analysis allows us to suggest a simple generalization of the experiment of Simmons *et al.*[8] which will permit the direct experimental verification of the fractional statistics of the quasiparticles in the fractional quantum Hall state.

Classical equations of motion.—The semiclassical theory starts with the classical equations of motion for the guiding-center coordinates which describe the motion of the particles in the lowest Landau level:[7]

$$\mathbf{0} = e\mathbf{r}_j \times \mathbf{B}/c - \nabla V(\mathbf{r}_j), \tag{2}$$

where $\mathbf{B} = B\hat{\mathbf{z}}$, and $V(\mathbf{r}_j)$ is the total potential energy at the location $\mathbf{r}_j$ of the jth particle, i.e., the sum of the external potential $U(\mathbf{r}_j)$ plus the interaction potential with all the other particles, $v(\mathbf{r}_j)$,

$$V(\mathbf{r}_j) = U(\mathbf{r}_j) + \sum_{k \neq j} v(\mathbf{r}_j - \mathbf{r}_k). \tag{3}$$

The equations of motion are independent of the particle statistics. The total potential energy is a constant of the motion. For noninteracting particles ($v=0$) each particle moves along an equal-potential contour with energy $U(r_j) = \varepsilon_j$. These equal-potential contours, and hence the classical trajectories, generically consist of closed orbits. Classical bound states occur in the neighborhood of local minima of the potential, which is unsurprising, and in the neighborhood of potential maxima. It is important to note that in the neighborhood of a given extremum of the potential, the classical bound states can be ordered such that trajectory j is strictly inside (closer to the extremum) than trajectory $j+1$; for a potential maximum this implies $\varepsilon_{j+1} < \varepsilon_j$.

Semiclassical quantization of noninteracting anyons.—The Bohr-Sommerfeld quantization condition for periodic orbits requires that the classical action is an integer multiple of h. Semiclassical quantization reproduces the Bohr-Sommerfeld condition but with an additive shift due to the phase of the fluctuation determinant.[13] In the large-B limit there are two contributions to the action: an Aharonov-Bohm contribution and a statistical contribution. These two contributions are similar in form since it is possible to treat[2] the fractional statistics as an Aharonov-Bohm interaction as if the particles were bosons with a flux $\phi_{\text{stat}} = (\theta^*/2\pi)\phi^*$ tied to each particle.

To begin with, we consider quantizing the classical bound states associated with a single extremum of the potential $U(r)$, which for concreteness we take to be a potential maximum. Since the particles are noninteracting, there is a separate quantization condition for each particle

$$2\pi n_j = 2\pi BA(\varepsilon_j)/\phi^* - (j-1)\theta^* - \theta_0, \tag{4}$$

where n_j is an integer, ε_j are the one-particle energies, $A(\varepsilon)$ is the area of the equal-potential contour of energy ε, $j-1$ is the number of particles enclosed by the trajectory of particle j, and $\theta_0 = \pi$ is the contribution to the phase from the fluctuation determinant [see Refs. 7(b) and 7(c)]. Thus, an N-anyon state is specified by a set of N occupied orbitals which satisfy Eq. (4). This is our principle theoretical result. The total energy E of the N-anyon state is simply the sum of the single-particle energies

$$E = \sum \varepsilon_j. \tag{5}$$

We have derived Eq. (4) using the Bohr-Sommerfeld quantization condition with a correction term θ_0 included to obtain the correct semiclassical result for the single-particle problem. In a forthcoming paper, we will show that Eq. (4) can be rederived from an asymptotic analysis of certain exactly solvable cases.

To check the accuracy of the semiclassical quantization prescription, we compare the results with other known results. First, we note that for fermions, θ^* is 2π and hence the statistical interaction can be absorbed into a redefinition of n_j, and for bosons $\theta^* = 0$ so the statistical interaction is nonexistent; in both cases the statistical interaction results in no shift in the one-particle states. This is a correct and remarkable feature of noninteracting fermions and bosons. Second, we compute the maximum density of anyons that can be accommodated in a region of space in the lowest Landau level. To compute this at the semiclassical level, we consider the state ψ_0 with $n_j = 0$ for all j. Since $n_j \geq 0$ must be a nondecreasing sequence in order that ε_j be an increasing sequence, it follows that ψ_0 is the highest-density state allowed by the quantization conditions. Its density in units of the flux density is

$$\nu_m = \frac{N\phi_0}{BA(\varepsilon_N)} = \frac{2\pi N}{(e/e^*)[\pi + (N-1)\theta^*]}. \tag{6}$$

The exact result is[14] $\nu_m = (e^*/e)(2\pi/\theta^*)$, in agreement with Eq. (6) for large N.

Periodicity of the bound-state spectrum.—Consider the bound states in the vicinity of a single extremum of the potential and imagine trying to add a particle at a fixed energy E_F. As we vary the magnetic field, there will occur a series of special "resonant" values of the magnetic field at which this is possible, just as in the one-particle case summarized in Eq. (1). We wish to compute the magnetic-field interval between successive resonances.

To begin with, consider the case in which initially the system is in an $(N-1)$-particle state in which all the particles are bound to potential contours that lie inside the contour with $U(r)=E_F$. In this case, it is possible to add one more quasiparticle with energy E_F if and only if there is a solution to Eq. (4) with $j=N$ and $\varepsilon_j=E_F$. If this condition is satisfied for some value of the magnetic field B and particle number N, then it will also be satisfied for another value of B' and N', where

$$B-B'=(N-N')\left(\frac{\theta^*\phi^*}{2\pi A(E_F)}\right)+n\left(\frac{\phi^*}{A(E_F)}\right) \quad (7)$$

and n is an integer. Note that for quasiparticles in the fractional quantum Hall state, $\theta^*\phi^*/2\pi=\theta_0$, so the first term has period ϕ_0/A while the second has period ϕ^*/A; *if the number of particles is held fixed, the effective flux quantum determines the periodicity, while if the number of particles is allowed to vary, the bare flux quantum determines the periodicity.*

In the more general $(N-1)$-particle state, not all the particles lie inside the contour $U(r)=E_F$. If some particles are in orbits which enclose this contour, the complete set of coupled equations (4) and (5) must be solved since the addition of a particle to this contour shifts the energies of all one-particle states which enclose it.

Perturbative effect of interactions.—The classical dynamics of many interacting particles is quite complicated, even if they obey guiding-center equations of motion, Eq. (2). We can, however, incorporate the effect of weak interactions in first-order perturbation theory since to this order the interactions do not change the states, simply the energy. In this case, the quantization condition, Eq. (4) remains unchanged but the many-body energy becomes

$$E=\sum_j \varepsilon_j+\sum_{i>j} v_{ij}\,, \quad (8)$$

where v_{ij} is the interaction energy[15] between particles i and j. The energy to add one particle in the outermost orbit remains ε_j, even in the presence of these interactions, so Eq. (7) remains valid.

Application to experiments in narrow quantum Hall devices.—It was noted in Ref. 9 that the longitudinal resistance ρ_{xx} of a narrow quantum Hall device can be used as a spectroscopic probe of the quasiparticle bound states in the bulk of the sample. In narrow samples, the Hall current is carried by edge states, and ρ_{xx} is determined by the interedge scattering rate. So long as the distance between edges is large compared to the Landau length, the scattering rate is astronomically low; $\rho_{xx}\approx 0$. If there is a weakly localized state in the bulk with energy near the Fermi energy and with an extent comparable to the width of the sample, scattering between edges through the localized state is far more likely than the direct process. As the magnetic field is varied, the energy of the localized states changes according to Eq. (4), and hence a periodic variation of the bound-state spectrum would be expected to produce a corresponding periodic variation in ρ_{xx}.

In Refs. 16 and 9, we considered the possibility of resonant scattering, in which a quasiparticle scatters coherently from one edge to the other, and only makes a virtual transition to the localized state. As discussed in Ref. 16, when resonant scattering is the dominant process, one should observe very sharp peaks in ρ_{xx} as a function of B whenever the resonance condition is satisfied. In the data of Simmons *et al.*, rather sharp peaks are observed at the edge of the $\nu=2$ plateau, which may be due to resonant tunneling. In the rest of the data, the periodic structure is less sharply peaked and is difficult to reconcile with resonant tunneling.

An alternative source of oscillatory behavior as a function of the bound-state spectrum is provided by the process pictured in Fig. 1(a). Here, we show a picture of a narrow quantum Hall device in cross section. The left edge states carry current in the direction out of the page and the right edge states carry current in the opposite direction. There is a net current due to the fact that the chemical potential on the left μ_L is greater than that on the right μ_R. In the bulk of the channel, there is a potential mountain of height $V_{\max}$, also seen in cross section; the solid dots indicate the positions of the quantized orbits for possible single quasihole states bound to this mountain. Without loss of generality, we have chosen the mountain to be closer to the left edge of the sample than to the right, so on the time scales of interest the bound states associated with this mountain are in chemical equilibrium with the left edge; at temperature $T=0$ there are essentially no holes in bound states with energy

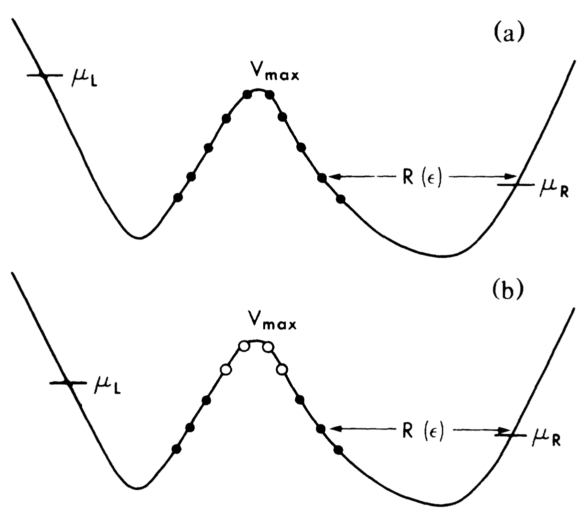

FIG. 1. Cross-sectional views of a Hall channel with a potential mountain near the left edge. The solid circles correspond to quantized single-particle bound states that are occupied by quasiholes and the open circles are unoccupied states. The situation (a) in which $\mu_L>V_{\max}>\mu_R$ and (b) in which $V_{\max}>\mu_L>\mu_R$ and $N_T=2$.

less than μ_L, which we have taken to be greater than the height of the potential mountain. The rate-limiting process which determines ρ_{xx} is tunneling from the localized state to the right edge. Alternatively, this process can be described as tunneling of a hole from the right edge to the bound state. This can occur at any energy greater than μ_R for which a quantized bound state exists, but since the probability decays exponentially with distance between the bound state and the edge $R(\varepsilon)$, it is clear from the picture that $R(\varepsilon) > R(\mu_R)$ for $\varepsilon > \mu_R$, so the process is dominated by the states with energy closest to μ_R. Since, by the same token, the equilibration rate of the bound states with the left edge is much faster than with the right edge, only the zero- and one-quasihole bound states are relevant. Thus, we assume $\Delta E \gg h/\tau_L \gg h/\tau_R$, where ΔE is the spacing between bound states and h/τ_R and h/τ_L are the tunneling rates from the left and right edges, respectively. In this case, the $1/\tau_L$, and hence ρ_{xx}, depend on $R(\varepsilon_R)$, where ε_R is the energy of the quantized bound state with energy closest to μ_R; therefore ρ_{xx} will have periodic structure with period determined by Eq. (1) with $E_F = \mu_R$.

The situation is richer if we consider a system with the same geometry, but with a somewhat smaller value of μ_L, as shown in Fig. 1(b). Again, assume that, on the relevant time scales, the bound states are in chemical equilibrium with the left edge states. Thus, ρ_{xx} is determined by the transition rate between quasihole states which consist of a fixed number N_T of quasiholes near the top of the mountain associated with potential contours with energies greater than μ_L, and zero or one quasihole in a bound state with energy near μ_R. In this situation, three sorts of effects should be observable in the variation of ρ_{xx} as a function of B: (1) ρ_{xx} should exhibit a periodic sequence of peaks with a short period

$$\Delta B_1 = \phi^*/2\pi A(\mu_R) \tag{9a}$$

reflecting the variation of the quantization condition on the quasiparticle bound state with energy $\sim\mu_R$ at fixed N_T. The period is short because $A(\mu_R)$ is relatively large; as before, this effect is a measure of the fractional charge. (2) Because of the long-range nature of the statistical interaction, a shift in the number of quasiholes near the mountain top will produce a shift in the large quantized orbits with energy near μ_R; when $N_T \rightarrow N_T + 1$, the pattern of peaks shifts by a fraction $\theta^*/2\pi$ of the period ΔB_1. According to Eq. (4), N_T will increase by $N_T \rightarrow N_T + 1$ every time the flux through the potential contour with energy μ_L increases by $\phi^*\theta^*/2\pi = \phi_0$. Thus, there should appear sudden shifts in the phase of the periodic variation of ρ_{xx} every time B changes by

$$\Delta B_2 = \phi_0/2\pi A(\mu_L) , \tag{9b}$$

which is much longer than ΔB_1 since $A(\mu_L) \ll A(\mu_R)$. (3) The addition of $2\pi/\theta^*$ quasiparticles to the top of the mountain ($N_T \rightarrow N_T + 2\pi/\theta^*$) results in a shift of the pattern of peaks by a full period, ΔB_1. ρ_{xx} should therefore exhibit a second period, $\Delta B_3 = (2\pi/\theta^*)\Delta B_2$. *Observation of either of these latter two effects would constitute the first direct measurement of fractional statistics.* Finally, in real systems, there should be an effect of the direct Coulomb interaction between the particles at the top of the mountain and the large bound state, which will be small so long as $A(\mu_L)$ is sufficiently large. This results in a small change in the bound-state spectrum whenever N_T changes by 1 and a slight deviation from perfect periodicity with the longer interval, ΔB_3.

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[15] The interaction energy is given by

$$v_{ij} = \oint_{U(r_i)=\varepsilon_i} \frac{dr_i}{L_i} \oint_{U(r_j)=\varepsilon_j} \frac{dr_j}{L_j} \frac{e^2B^2 v(\mathbf{r}_i - \mathbf{r}_j)/c^2}{|\nabla U(\mathbf{r}_i)||\nabla U(\mathbf{r}_j)|} ,$$

where L_j is the length of the jth orbit.

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