# Conditional Averaging Procedure for the Elimination of the Small-Scale Modes from Incompressible Fluid Turbulence at High Reynolds Numbers

W. D. McComb and A. G. Watt
*Department of Physics, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom*
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High-wave-number modes are eliminated in a band characterized by its width parameter $\lambda$ in $k$ space. The requisite conditional average is evaluated as an approximation in which coupling effects are neglected to order $\lambda^2$, for small $\lambda$. A fixed point was found under renormalization-group transformation, which corresponded to the Kolmogorov "$-\frac{5}{3}$" spectrum with spectral constant $\alpha = 1.60$.

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The work to be described here has its roots in the method of *iterative averaging*, as reported previously.[1,2] However, the present method will be presented as complete in itself. We shall begin by formulating and stating the problem.

Consider the turbulent velocity field in wave-number space $u_\alpha(\mathbf{k},t)$, on the interval $0 \le k \le k_0$, with $k_0$ being defined through the dissipation integral

$$\epsilon = \int_0^\infty 2\nu_0 k^2 E(k)\,dk \simeq \int_0^{k_0} 2\nu_0 k^2 E(k)\,dk\,, \tag{1}$$

where $\epsilon$ is the dissipation rate, $\nu_0$ is the kinematic viscosity, and $E(k)$ is the energy spectrum. This definition ensures that $k_0$ is of the same order of magnitude as the Kolmogorov dissipation wave number.

As is usual, we denote a global averaging operation by Dirac brackets, thus $\langle\,\rangle$. We restrict our attention to incompressible fluids subject to a chaotic velocity field with zero mean, and consider only fields that are homogeneous, isotropic, and stationary in time. As a result, the second-order moment takes the following form:

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle = Q(k,t-t')D_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\,, \tag{2}$$

where $\alpha,\beta = 1$, 2, or 3, and $Q(k,t-t')$ is the spectral density. The projector $D_{\alpha\beta}(\mathbf{k})$ arises because of the incompressibility condition and is given by

$$D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\mathbf{k}|^{-2} \tag{3}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. The energy spectrum is introduced by taking $t = t'$ and setting $Q(k,0) = Q(k)$; thus,

$$E(k) = 4\pi k^2 Q(k)\,. \tag{4}$$

In order to introduce the renormalization-group approach, we divide up the velocity field at $k = k_1$ in the following way:

$$u_\alpha(\mathbf{k}) = \begin{cases} u_\alpha^-(\mathbf{k}) & \text{for } 0 \le k \le k_1\,, \\ u_\alpha^+(\mathbf{k}) & \text{for } k_1 \le k \le k_0\,, \end{cases} \tag{5}$$

where $k_1$ is defined by

$$k_1 = (1-\lambda)k_0\,, \tag{6}$$

with the bandwidth parameter $\lambda$ satisfying the condition $0 \le \lambda \le 1$.

In principle, the renormalization-group approach now involves two stages: (A) Solve the Navier-Stokes equation (NSE) on $k_1 \le k \le k_0$. Substitute that solution for the mean effect of the high-$k$ modes into the NSE on $0 \le k \le k_1$. This results in an increment to the viscosity: $\nu_0 \to \nu_1 = \nu_0 + \delta\nu_0$. (B) Rescale the basic variables, so that the NSE on $0 \le k \le k_1$ looks like the original Navier-Stokes equation on $0 \le k \le k_0$.

This procedure is appealingly simple and has a clear physical interpretation. But, as is well known, it has not proved easy to put it into practice in the turbulence problem. The aim of this Letter is to introduce a rational way of doing this, by an approximation in which the bandwidth $\lambda$ plays the part of a small parameter. We begin by introducing a conditional average which averages out the effect of high-$k$ modes, while keeping the $\mathbf{u}^-$ constant. That is, it is an average over the turbulent ensemble in which realizations are chosen to hold $\mathbf{u}^-$ constant. We represent it by the operator $A[\mathbf{u}^+|\mathbf{u}^-]$ and denote its effect, on the first shell of wave numbers to be eliminated, by $\langle\,\rangle_0$; thus,

$$A[\mathbf{u}^+|\mathbf{u}^-]u_\alpha u_\beta \cdots u_\gamma = \langle u_\alpha u_\beta \cdots u_\gamma\rangle_0\,. \tag{7}$$

It then follows from the definition that this operator, when acting on the low-$k$ modes, has the following properties:

$$\langle u_\alpha^-(\mathbf{k})\rangle_0 = u_\alpha^-(\mathbf{k})\,, \tag{8}$$

$$\langle u_\alpha^-(\mathbf{j})u_\beta^-(\mathbf{k}-\mathbf{j})\rangle_0 = u_\alpha^-(\mathbf{j})u_\beta^-(\mathbf{k}-\mathbf{j})\,. \tag{9}$$

We now wish to evaluate averages of this kind over the high-$k$ modes and express them in terms of global mean quantities. The problem we face is that the $\mathbf{u}^+$ field is not independent of the $\mathbf{u}^-$ field which we are holding constant. The two fields are, of course, coupled together through the nonlinear term in the Navier-Stokes equation. We tackle this difficulty by writing the high-$k$ modes in terms of a new field $\mathbf{v}^+$; thus

$$u_\alpha^+(\mathbf{k},t) = v_\alpha^+(\mathbf{k},t) + \Delta_\alpha^+(\mathbf{k},t)\,. \tag{10}$$

Here $\mathbf{v}^+$ is a field of the same general type as $\mathbf{u}^+$ and has the same properties under *global* averaging; thus,

$$\langle v_\alpha^+(\mathbf{k},t)\rangle = 0\,, \tag{11}$$

$$\langle v_\alpha^+(\mathbf{k},t)v_\beta^+(\mathbf{k}',t)\rangle = \langle u_\alpha^+(\mathbf{k},t)u_\beta^+(\mathbf{k}',t)\rangle\,. \tag{12}$$

However, the essential feature of $\mathbf{v}^+$ is that it is not coupled to the $\mathbf{u}^-$ modes. Thus, through Eq. (10) we introduce the function $\mathbf{\Delta}^+$ to take account of such mode coupling. Its properties under global averaging are as follows. From the condition of zero mean field, we have

$$\langle \Delta_\alpha^+(\mathbf{k},t)\rangle = 0\,, \tag{13}$$

while, from Eq. (12), we have

$$\langle \Delta_\alpha^+(\mathbf{k},t)\Delta_\beta^+(\mathbf{k}',t)\rangle = 0\,. \tag{14}$$

Note that this result also reflects the fact that global averaging destroys the coupling between different wave numbers for homogeneous fields; see Eq. (2).

With all these points in mind, we complete our specification of the two new fields by stating their properties under *conditional* averaging as

$$A[\mathbf{u}^+|\mathbf{u}^-]\mathbf{v}^+(\mathbf{k},t) = \langle \mathbf{v}^+(\mathbf{k},t)\rangle_0 = \langle \mathbf{v}^+(\mathbf{k},t)\rangle = 0 \tag{15}$$

and

$$A[\mathbf{u}^+|\mathbf{u}^-]\mathbf{\Delta}^+(\mathbf{k},t) = \langle \mathbf{\Delta}^+(\mathbf{k},t)\rangle_0 = O(\lambda^m)\,, \quad m \geq 1\,. \tag{16}$$

Now the equation of motion for incompressible fluid flow is the Navier-Stokes equation. It may be written in its spectral form as

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t)\,, \tag{17}$$

where

$$M_{\alpha\beta\gamma}(\mathbf{k}) = (2i)^{-1}[k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})]\,, \tag{18}$$

and $D_{\alpha\beta}(\mathbf{k})$ is given by Eq. (3). We take the arbitrary stirring forces $f_\alpha(\mathbf{k},t)$ to satisfy the usual requirements for a well-posed problem. That is, their direct effect is only felt at very small values of the wave number. Apart from that, we shall not specify them as, from our point of view, their only importance lies in their maintaining the stationarity of the velocity field against the viscous dissipation of energy.[3]

Now let us form the evolution equations for the explicit scale modes ($\mathbf{u}^-$) and the implicit scale modes ($\mathbf{u}^+$). We use Eqs. (5), (8), and (9) to transform the NSE into the key equation for the explicit scales:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) - \left\langle M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\right\rangle_0 = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t) + H_\alpha(\mathbf{k},t)\,. \tag{19}$$

The same procedure, and subtraction of (19) from (17), results in the equation for the implicit scales:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^+(\mathbf{k},t) = 2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^-(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)$$
$$+ M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j[u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t) - \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0] - H_\alpha(\mathbf{k},t)\,. \tag{20}$$

In both cases, $H_\alpha(\mathbf{k},t)$ is a correction which may be expressed in terms of the conditionally averaged mode couplings only, by using Eq. (10), along with Eqs. (15) and (16), to show that

$$H_\alpha(\mathbf{k},t) = 2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^-(\mathbf{j},t)\langle \Delta_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 - L_0\langle \Delta_\alpha^+(\mathbf{k},t)\rangle_0\,, \tag{21}$$

where $L_0 = [\partial/\partial t + \nu_0 k^2]$, thus indicating that the correction term is of order $\lambda^m$.

In order to complete the elimination of the high-$k$ modes from Eq. (19), we need to obtain an explicit expression for $\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$. We obtain an evolution equation for this quantity from (20), and it is readily shown that this takes the form

$$\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = \int_{-\infty}^{t} dt' \exp[(-\nu_0 j^2 - \nu_0|\mathbf{k}-\mathbf{j}|^2)(t-t')]M_{\beta\delta\epsilon}(\mathbf{j})$$
$$\times \int d^3p[4u_\delta^-(\mathbf{p},t')\langle u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 + 2\langle u_\delta^+(\mathbf{p},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0]$$
$$- 2\langle\langle u_\delta^+(\mathbf{p},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')\rangle_0 u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 - 2\langle u_\gamma^+(\mathbf{k}-\mathbf{j},t')H_\beta(\mathbf{j},t')\rangle_0\,. \tag{22}$$

It should be noted that this solution contains the triple conditional moment $\langle \mathbf{u}^+\mathbf{u}^+\mathbf{u}^+\rangle_0$, and that we can solve Eq. (20) for this in terms of the quadruple conditional moment $\langle \mathbf{u}^+\mathbf{u}^+\mathbf{u}^+\mathbf{u}^+\rangle_0$, and so on. Hence, the moment closure problem is still with us, although it can be argued that it has to some extent been tamed, as we have restricted it to a narrow band of wave numbers located at the end of the dissipation range.

We now make use of Eqs. (8)-(16) to evaluate conditional moments of the $\mathbf{u}^+$ in terms of the unconditional moments of the $\mathbf{v}^+$. At the same time, we solve (20) iteratively and show that the solution for $\langle \mathbf{u}^+\mathbf{u}^+\rangle_0$, to all orders of unconditional moments, is linearly dependent on $\mathbf{u}^-(\mathbf{k},t)$. With the procedures discussed above, we can derive an expression for the increment $\delta\nu_0$ as the sum of an infinite series in the unconditional moments of the $\mathbf{v}^+$ field.

Further details of these procedures will be given in a later paper, but, for the present, the essential point is that this elimination of the first shell of modes $(1-\lambda)\times k_0 < k < k_0$ is a quite general, rigorous method. However, in order to carry out the second step which completes one iteration of the renormalization group, we must make three approximations. As two of these are fixed in form by what are basically practical considerations, we shall deal with these first in order to isolate the essential approximation which lies at the heart of the present method.

First, we truncate the unconditional moment expansion in the $\mathbf{v}^+$ field at the lowest nontrivial order. This means that we consider only terms which are second order in the interaction strength (i.e., of superficial order $M^2$).

Second, the time integrations on the right-hand side of Eq. (22) are evaluated on the basis that the $\mathbf{u}^-$ are slowly varying on the time scales characteristic of the $\mathbf{u}^+$ (or, more pertinently, the $\mathbf{v}^+$). This is an example of a Markovian approximation.

Then, with all these points in mind, Eqs. (19)-(22) yield for the viscosity acting on the explicit scales:

$$\nu_1 = \nu_0 + \delta\nu_0\,, \tag{23}$$

where the formula for the increment to viscosity is

$$\delta\nu_0(k) = \frac{1}{k^2}\int d^3j\,\frac{L(\mathbf{k},\mathbf{j})Q_v^+(|\mathbf{k}-\mathbf{j}|)}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2} + O(\lambda^m)\,, \tag{24}$$

with $0 \le k \le k_1$, $k_1 \le j$, $|\mathbf{k}-\mathbf{j}| \le k_0$, and $Q_v^+$ is merely an extension of the spectral density as defined by (2) to the $\mathbf{v}^+$ field. The coefficient $L(\mathbf{k},\mathbf{j})$ is given by

$$\begin{aligned} L(\mathbf{k},\mathbf{j}) &= -2M_{\rho\beta\gamma}(\mathbf{k})M_{\beta\rho\delta}(\mathbf{j})D_{\delta\gamma}(|\mathbf{k}-\mathbf{j}|) \\ &= -\frac{[\mu(k^2+j^2)-kj(1+2\mu^2)](1-\mu^2)kj}{k^2+j^2-2kj\mu}\,, \end{aligned} \tag{25}$$

where $\mu$ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{j}$.

Now let us consider our principal approximation. In order to make a specific calculation, we shall assume that the coupling between distinct Fourier modes is local in wave number. That is, we shall assume that $\lambda$ is large enough for $\mathbf{u}(\mathbf{k}_0)$ to be independent of $\mathbf{u}(\mathbf{k}_1)$, and at the same time that $\lambda$ is small enough for us to represent the Fourier components in the band by means of a first-order Taylor series. In this way, we impose both upper and lower bounds on $\lambda$, when we make the identification

$$v_\alpha^+(\mathbf{k},t) = u_\alpha^+(\mathbf{k}_0,t) + (\mathbf{k}-\mathbf{k}_0)\cdot\nabla u_\alpha^+(\mathbf{k},t)|_{k=k_0} + O(\lambda^2)\,. \tag{26}$$

Note that we conclude that terms of order $\lambda^2$ have been neglected because the maximum value of $|\mathbf{k}-\mathbf{k}_0|$ is $\lambda$, and hence $\Delta_\alpha^+$ and $H_\alpha$ are also both $O(\lambda^2)$.

We extend the procedure to further wave-number shells as follows: (a) Set $u_\alpha^-(\mathbf{k},t) = u_\alpha(\mathbf{k},t)$ in Eq. (19), so that we now have a new NSE with effective viscosity $\nu_1(k)$ for Fourier modes on the interval $0<k<k_1$. (b) Make the decomposition of (5), but this time at $k=k_2$, such that $u_\alpha^+(\mathbf{k},t)$ is now defined in the band $k_2 \le k \le k_1$. (c) Repeat the procedures used to eliminate the first shell of modes in order now to eliminate modes in the band $k_2 \le k \le k_1$.

In this way, we can progressively eliminate the effect of high wave numbers in a series of bands $k_{n+1} < k < k_n$, where

$$k_n = (1-\lambda)^n k_0,\quad 0 \le \lambda \le 1\,, \tag{27}$$

with, by induction, the recursion relation for the effective viscosity given by

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k)\,, \tag{28}$$

where the increment of order $n$ takes the form

$$\delta\nu_n(k) = \frac{1}{k^2}\int d^3j\,\frac{L(\mathbf{k},\mathbf{j})\{Q(l)|_{l=k_n} + (l-k_n)\partial Q(l)/\partial l|_{l=k_n} + O(\lambda^2)\}}{\nu_n j^2 + \nu_n|\mathbf{k}-\mathbf{j}|^2}\,. \tag{29}$$

Also, we may form an energy equation for the explicit scales, hence obtaining the renormalized dissipation relation, viz.,

$$\int_0^{k_n} 2\nu_n(k)E(k)dk = \epsilon\,, \tag{30}$$

which may be compared with the unrenormalized form given in Eq. (1).

If we now assume that the energy spectrum in the band is given by a power law and make the scaling transformation

$$k_{n+1} = hk_n\,, \tag{31}$$

where, for compactness, we define $h=(1-\lambda)$, it follows from Eqs. (28) and (29) that the effective viscosity may be written as

$$\nu(k_n k') = \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(k')\,, \tag{32}$$

where $\alpha$ is the constant of proportionality in the assumed spectrum. Now the recursion relation becomes

$$\tilde{\nu}_{n+1}(k') = h^{4/3}\tilde{\nu}_n(hk') + h^{-4/3}\delta\tilde{\nu}_n(k') \tag{33}$$

with

$$\delta\tilde{\nu}_n(k') = \frac{1}{4\pi k'^2}\int d^3j'\,\frac{L(\mathbf{k}',\mathbf{j}')Q'}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n(hl')l'^2} \tag{34}$$

for the wave-number bands $0 \le k' \le 1$, $1 \le j'$, $l' \le h^{-1}$,

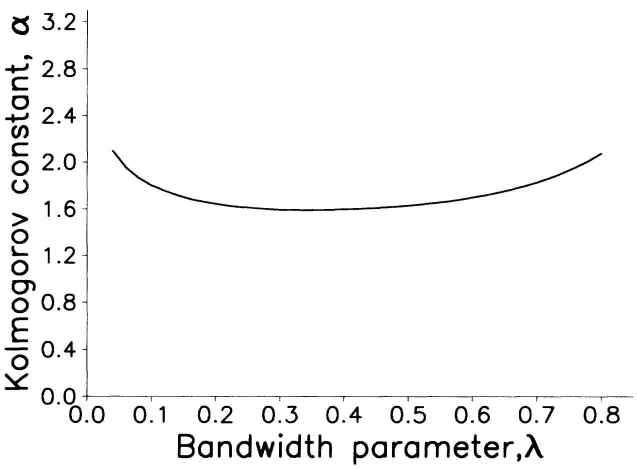


FIG. 1. Dependence of the calculated value of the Kolmogorov spectral constant on the choice of width of band in which modes were eliminated.

where $l' = |\mathbf{k}' - \mathbf{j}'|$, and

$$Q' = h^{11/3} - \tfrac{11}{3} h^{14/3}(l' - h^{-1}) + \cdots \quad (35)$$

($\cdots$ denotes higher-order terms). Iteration of Eqs. (33) and (34) reaches a fixed point and details of this calculation will be given in a further communication, but, for the present we note that once a fixed point is found we can calculate the Kolmogorov constant by solving Eqs. (30) and (32) simultaneously. In Fig. 1 we plot the calculated values of $\alpha$ against the bandwidth parameter $\lambda$. One merit of taking $\alpha$ as a test is that it does have known experimental values, albeit scattered in the range $1.2 < \alpha < 2.2$. From the figure, it can be seen that our calculated value of the Kolmogorov spectral constant is $\alpha = 1.60$, in good agreement with experiment, for the range of $\lambda$ for which the theory is valid. At large values of $\lambda$, one may observe the breakdown of the first-order Taylor series approximation, while at small values, one sees the effects of mode coupling, which would invalidate the assumption that $\mathbf{u}(\mathbf{k}_0)$ is independent of $\mathbf{u}(\mathbf{k}_1)$.

Before concluding, we make two points. First, an explicit operator with the properties of the conditional average set out in Eqs. (8), (9), and (15) has been obtained in approximate form as a variation between different realizations. Second, the form of the increment to the effective viscosity, as given by (24), is identical to the expression for the total effective viscosity which is obtained when time dependences in the direct interaction approximation are represented by exponential decays.[4]

We conclude by pointing out that the main effect of the present work is to recast the problem as: How does one make a physically reasonable and mathematically tractable choice which expresses $\mathbf{v}^+$ in terms of $\mathbf{u}^+$? Evidently Eq. (26) is a natural choice, but the effect of making other choices will be explored in further work.

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[1] W. D. McComb, Phys. Rev. A **26**, 1078 (1982).

[2] W. D. McComb, in *Direct and Large-Eddy Simulation*, edited by U. Schumann and R. Friedrich, Notes on Numerical Fluid Mechanics Vol. 15 (Vieweg, Braunschweig, 1986).

[3] It should, perhaps, be emphasized that we are dealing with turbulence generated by the Navier-Stokes equation, not the case of strongly stirred hydrodynamics.

[4] R. H. Kraichnan, Phys. Fluids **7**, 1163 (1964).