Hierarchy of Soliton Solutions to the Gauged Nonlinear Schrödinger Equation on the Plane

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We find static, self-dual solutions to the SU(N) gauged, nonlinear Schrödinger equation in two spatial dimensions with scalar matter in the adjoint representation. The gauge field is determined dynamically by the Chern-Simons interaction with nonrelativistic scalar matter. The hierarchy is associated with the Toda hierarchy, which can be solved explicitly. The $N=2$ case reduces to the Liouville equation found by Jackiw and Pi when looking at the nonlinear Schrödinger equation gauged by U(1).

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We will study in this paper the gauged, nonlinear Schrödinger equation in two space, $\mathbf{x}=(x_1,x_2)$, and one time, t, dimension for a complex scalar field, $\Phi(\mathbf{x},t)$, in the adjoint representation of the gauge group SU(N) with vector potential $(A_0,\mathbf{A})=(A_0,A_1,A_2)(\mathbf{x},t)$:

$$i\frac{\partial}{\partial t}\Phi=-\mathbf{D}^2\Phi+e[A_0,\Phi]+g[[\Phi^*,\Phi],\Phi]\,,\qquad D_i\Phi=\partial_i\Phi+e[A_i,\Phi]\,. \tag{1}$$

The system is gauge invariant under the gauge transformation by $g\in \mathrm{SU}(N)$,

$$A_\mu\to g^{-1}A_\mu g+g^{-1}\partial_\mu g\,,\quad \Phi\to g^{-1}\Phi g\,.$$

We have been motivated to study this system by the recent study of Jackiw and Pi[1] of a similar system with gauge group U(1) and the charge $[\Phi^*,\Phi]$ replaced by $\Phi^*\Phi$.

In recent years, enormous progress has been made in understanding integrable quantum field theories in two dimensions,[2] first in the classical realm, and then in the quantum. These models can often be understood in terms of hierarchies, such as the Korteweg–de Vries hierarchy or the Toda hierarchy.[3] The nonlinear Schrödinger equation in 1+1 dimensions for a complex singlet field has been solved both classically and quantum mechanically. It too is the beginning of such a hierarchy.

We consider the gauged nonlinear Schrödinger equation as building upon this work, following the dawning of a new age of three-dimensional models. On the one hand, there is three-dimensional gravity[4] and more generally the Chern-Simons theory,[5] which seem to be exactly integrable as quantum theories. On the other hand, there are only a few integrable classical theories in three dimensions: the Kadometsev-Petviashvili equation,[6] the Davey-Stewartson equation,[7] and now the gauged nonlinear Schrödinger equation first considered by Jackiw and Pi[1] is a new candidate.

In this work we will show that the gauged nonlinear Schrödinger equation, with a dynamically determined gauge field, is part of a hierarchy of equations for gauge groups SU(N), of which Jackiw and Pi considered the U(1) subgroup. The gauge field is dynamically determined by a convariantly conserved current J^μ. If

$$J^\mu=(\rho,\mathbf{J})=([\Phi^*,\Phi],-i(\Phi^*\mathbf{D}\Phi-\mathbf{D}\Phi^*\Phi))$$

so that $D_\mu J^\mu=0$, then the gauge field is dynamically determined by the equation

$$(k/2\pi)\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}=eJ^\mu\,, \tag{2}$$

where $F_{\alpha\beta}=[D_\alpha,D_\beta]$ and k is an integer. Equation (2) arises from a Chern-Simons interaction in the Lagrangian. We will find explicit static solutions of Eqs. (1) and (2), relying upon the SU(N) Toda hierarchy. The $N=2$ case is the Liouville equation; hence this reduces to the system considered by Jackiw and Pi. Each member of the hierarchy can be considered as the $k\to\infty$ limit of a topologically massive Yang-Mills theory in 2+1 dimensions, since the Chern-Simons term gives rise to a topological mass for the gauge field.[8] Equations (1) and (2) can be derived from the Lagrangian

$$L=\frac{k}{4\pi}\varepsilon^{\alpha\beta\gamma}\mathrm{Tr}(A_\alpha\partial_\beta A_\gamma+\tfrac{2}{3}A_\alpha A_\beta A_\gamma)+i\,\mathrm{Tr}\Phi^*\,\partial_t\Phi+e\,\mathrm{Tr}(\Phi^*[A_0,\Phi])-\mathrm{Tr}(\mathbf{D},\Phi)^2+(g/2)\mathrm{Tr}[\Phi^*,\Phi]^2\,. \tag{3}$$

We also consider the $N=\infty$ limit of the hierarchy. The Toda equations have nonlinear symmetries called W_N algebra[9] for the SU(N) Toda theory.[10] The generators of these symmetries appear in the explicit soliton solutions. The $N=\infty$ limit[11] is especially interesting because the W_∞ algebra is linear. The SU(∞) Toda equation can be derived from a three-dimensional theory of gravity called the Einstein-Weyl theory.[12] This theory has a propagating graviton, unlike the usual theory of gravity in three dimensions, which is purely topological.

To first consider the solutions, we follow Jackiw and Pi[1] in considering the Hamiltonian point of view, with Hamiltonian $H=\int d\mathbf{x}\mathcal{H}$,

$$\mathcal{H}=\mathrm{Tr}(\mathbf{D}\Phi)^2-(g/2)\mathrm{Tr}[\Phi^*,\Phi]^2, \tag{4}$$

plus the constraint (2). Then (1) is a consequence of the Schrödinger equation

$$i\frac{\partial\Phi}{\partial t}=\frac{\delta H}{\delta\Phi^*}. \tag{5}$$

The static solutions can be found by using the identity noted by Jackiw and Pi:

$$\mathrm{Tr}(\mathbf{D}\Phi)^2=\mathrm{Tr}[(D_1\pm iD_2)\Phi]^2\pm\mathrm{Tr}\boldsymbol{\partial}\times\mathbf{J}\pm e(\mathrm{Tr}\Phi^*\mathbf{B}\Phi). \tag{6}$$

Finding the static solutions is equivalent to minimizing the energy functional

$$H=\int d\mathbf{x}\,\mathrm{Tr}[(D_1\pm iD_2)\Phi]^2 \tag{7}$$

provided that $\int d\mathbf{x}\,\mathrm{Tr}\mathbf{D}\times\mathbf{J}=0$ and the couplings are fixed by $g=\mp 4\pi e^2/k$. The crucial observation of this paper is that the energy-minimizing equation

$$(D_1\pm iD_2)\Phi=0, \tag{8a}$$

along with the constraint

$$B=F_{12}=-(2\pi e/k)[\Phi^*,\Phi], \tag{8b}$$

can be obtained as a dimensional reduction of the four-dimensional self-dual equation

$$G_{\mu\nu}=\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}=\partial_\mu W_\nu-\partial_\nu W_\mu+[W_\mu,W_\nu], \tag{9}$$

with

$$W_{1,2}=A_{1,2},\quad W_3+iW_4=\Phi,\quad W_3-iW_4=\Phi^*,$$

where all fields W_μ satisfy $\partial_3 W_\mu=0=\partial_4 W_\mu$. If we call $z=x_1+ix_2$, $\bar z=x_1-ix_2$, then we can obtain solutions to (8a) and (8b) with the ansatz

$$A_z=\sum_a a^a H_a,\quad A_{\bar z}=\sum_a b^a H_a,$$
$$\Phi=\sum_a c^a E_a,\quad \Phi^*=\sum \bar c^a E_{-a} \tag{10}$$

for a Lie algebra with generators H_a, E_a, E_{-a} so that

$$[H_\alpha,H_\beta]=0,\quad [H_\alpha,E_{\pm\beta}]=\pm K_{\alpha\beta}E_{\pm\beta},$$
$$[E_\alpha,E_\beta]=\delta_{\alpha,-\beta}H_\beta, \tag{11}$$

where $K_{\alpha\beta}$ is the Cartan matrix $K_{\alpha\beta}=\alpha\beta/\alpha^2$ for simple roots α,β. We note that this is because B can be diagonalized by a gauge transformation. We can find solutions when A_z and $A_{\bar z}$ lie in the Cartan subalgebra generated by the H_a's. Φ,Φ^* take the form shown. However, we can solve this explicitly only for $c^a=\bar c^a$. Otherwise there is an unknown function in the problem. With the above ansatz, we can solve $D_{\bar z}\Phi=0$ and $D_z\Phi^*=0$ for a^a,b^a in terms of c^a. Then we can solve the constraint (8b) for $c^a=\exp\phi^a$ so that

$$\partial_z\partial_{\bar z}\phi^a=-\frac{2\pi e}{k}\sum_\beta K_{a\beta}e^{\phi^\beta}, \tag{12}$$

which is the Toda molecule equation.[5,10,13] For SU(2), one obtains the Liouville equation, as in Ref. 1. This is expected, since we have made an Abelian ansatz for the SU(2) gauge fields.

For SU(3), we obtain coupled equations

$$\partial_z\partial_{\bar z}\phi^1=(-2\pi e/k)(2e^{\phi^1}-e^{\phi^2}), \tag{13a}$$

$$\partial_z\partial_{\bar z}\phi^2=(-2\pi e/k)(2e^{\phi^2}-e^{\phi^1}). \tag{13b}$$

Let $-2\pi e/k=1$ henceforth.

The Toda equations have been solved explicitly and elegantly by Leznov and Saveliev.[14] For SU(N),[10] one can solve for the ϕ^a's in terms of $2(N-1)$ holomorphic functions $f_i^\pm, i=1,\ldots,N-1$,

$$e^{-\phi^i}=\chi\chi^1\cdots\chi^{(i-1)}\psi\psi^1\cdots\psi^{(i-1)}, \tag{14}$$

$$\psi=\psi(z)=\sum_{a=1}^N\psi_a(z)\xi^a, \tag{15a}$$

$$\chi=\chi(\bar z)=\sum_{a=1}^N\chi_a(\bar z)\frac{\partial}{\partial\xi_a}; \tag{15b}$$

$$[\xi^a,\xi^b]_+=0=\left[\frac{\partial}{\partial\xi^a},\frac{\partial}{\partial\xi^b}\right]_+$$

denotes differentiation. The $\psi_a(z)$'s denote a column vector

$$\prod_{j=1}^{N-1}f_j^+(z)^{-G_{ij}}M_+(z)|\lambda_1\rangle \tag{16a}$$

and $\chi_a(\bar z)$ denotes the vector

$$\langle\lambda_1|M^-(\bar z)^{-1}\prod_{j=1}^{N-1}f_j^-(\bar z)^{-G_{ij}}, \tag{16b}$$

where G_{ij} is the inverse Cartan matrix and $M^\pm$ satisfy the differential equations

$$\partial_z M^+=M^+\sum_{j=1}^{N-1}f_j^+(z)E_{-j}, \tag{17a}$$

$$\partial_{\bar z} M^-=M^-\sum_{j=1}^{N-1}f_j^-(\bar z)E_{+j}, \tag{17b}$$

and λ_i are the $N-1$ fundamental weight vectors $\lambda_i\alpha_j/\alpha_j^2=\delta_{ij}$.

For SU(2) this reduces to the solution used by Jackiw and Pi if we set $f_+=f_-$, while for SU(3) we have

$$e^{-\phi^1}=\frac{|f_1(z)|^{-1/3}|f_2(z)|^{-2/3}}{[1+|\int f_1(z)|^2+|\int f_2(z)\int^z f_2(z')|^2]^{-1}}, \tag{18a}$$

$$e^{-\phi^2}=e^{-2\phi^1}\partial_z\partial_{\bar z}\phi^1. \tag{18b}$$

If we choose $f_1(z)=1$, $f_2(z)=z^n$, we have for the charge distribution

$$[\Phi^*,\Phi]=e^{2\phi^1}H_1+e^{2\phi^2}H_2\,,$$

where

$$e^{2\phi^1}=r^{2n/3}\left(1+r^2+\frac{r^{2(n+2)}}{(n+1)(n+2)}\right)^{-2}.$$

Equation (18) for $e^{-\phi^i}$ is very illuminating. For the case of the Liouville equation, it is essentially the Bäckland transformation which generates solutions in terms of free fields. For $N=2$,

$$e^{-\phi^1}=\sum\psi_a(z)\chi_a(\bar z),\quad e^{-\phi^2}=1\,, \tag{19}$$

with

$$(-\partial_z^2+T_{zz})\psi(z)=0=(-\partial_{\bar z}^2+T_{\bar z\bar z})\chi(\bar z)\,,$$

where $T_{zz},T_{\bar z\bar z}$ are components of the energy-momentum tensor. For general N,

$$[-\partial_z^N+U_{(2)}\partial^{N-2}+\cdots+U_{(N)}]\psi_a(z)=0 \tag{20}$$

with a corresponding equation for χ. One can construct from the $U_{(k)}$'s conformal fields of conformal spin k, starting with the energy-momentum tensor $U_{(2)}$.[10] These fields generate the classical W algebras that are nonlinear symmetries of the Toda equations. The fields can be expressed in terms of kth-order differentials on the $(\mathbf{A},\Phi,\Phi^*)$ fields as seen in the formula

$$U_{(d)}(z)=e^{\phi_1}\left(\partial_z^d-\sum_{i=2}^{d-1}U_{(i)}(z)\partial_z^{d-i}\right)e^{-\phi_1}. \tag{21}$$

It has been noted that since the Cartan matrix has the form $K_{-\alpha\beta}=2\delta_{\alpha\beta}-\delta_{\alpha,\beta-1}-\delta_{\alpha,\beta+1}$, one can formally take the $N\to\infty$ limit so that $K_{\alpha\beta}\phi_\beta\to d^2\phi(t)^{(10)}/dt^2$ for a continuum variable t. This limit of the Toda equations can be obtained from Einstein-Weyl gravity.[12] While the pure Chern-Simons theory is topological, when it is coupled to a scalar field in curved space it ceases to be so. A similar statement can be made about the relation between Einstein-Hilbert gravity and Einstein-Weyl gravity with the conformal factor replacing the scalar field.

Finally, we list some additional points worthy of further investigation. First, one might consider complexifying time to make the theory four dimensional. One could pursue in this direction a possible relation to four-dimensional self-dual gravity.[15] (Einstein-Weyl gravity can in fact be derived from self-dual gravity by a quotient by its Killing vector.) Second, the vortices obtained in this theory are non-Abelian vortices, which after quantization should have non-Abelian fractional statistics. One can quantize the Toda molecule equation[16] and obtain an R matrix, which in fact describes such statistics. The scalar fields $\Phi=\sum e^{\phi^a}E_a$ act as vertex operators. It is interesting to investigate this further in relation to the quantum Hall effect as well as vortex scattering. Last,[17] this investigation may shed light on the relation of the Donaldson invariants for four-dimensional topological Yang-Mills theory (i.e., integrals of differential forms on the moduli space of instantons) and the Vaughn-Jones invariants as computed by the holonomy around a knot of the flat connection in the Chern-Simons gauge theory.

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